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- Surfaces of constant negative curvature
- Mathematical models in biology from point of view of Lie group analysis
- Group analysis of a tumour growth model

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Surfaces of constant negative curvature

Translation of Bäcklund’s Paper

"Om ytor med konstant negativ krökring"

By Ilir Berisha

M.Sc. Thesis in Mathematics, March 2005

Blekinge Institute of Technology
Department of Mathematics and Science
Supervisor: Prof. Nail H. Ibragimov

[Albert Victor Bäcklund, Om ytor med konstant negativ krökring, Lunds Universitets Års-skrift T. XIX. p. 1-48]

Abstract. The work is a translation from Swedish into English of Bäcklund’s famous paper on non-point transformations known in the literature of Bäcklund transformations. These transformations are widely used in Soliton theory (see [1], [2]).

Bäcklund has considered in this paper how to derive new surfaces of constant curvature. He characterized it as analytic, geometric and as a transformation from a domain of the space to another. He gives a generalization of surfaces with constant curvature for Bianchi’s transformation, and he works with determining of geodesic lines of the surfaces of constant negative curvature. He shows that, starting with a surface of constant curvature with known geodesic lines, one can obtain an infinite number of other surfaces of the same curvature without integrating any differential equation. These surfaces can be obtained performing only quadratures.

1 Introduction

According to Bianchi we can obtain an infinite number of surfaces with the same curvature from a surface of constant negative curvature. Bianchi described his method in 1879 in [Giornale di Matematiche, pubblicato da G. Battaglini T. XVII] and more detailed in a paper which was printed the same year in Pisa and titled "Ricerche sulle superficie una curvatura costante e sulle elicoidi", and in a shorter paper in 1880 in [volume XVI of Mathematische Annalen]. In "Ricerche etc." and in "Mathem.Ann." Bianchi determined the nature of those surfaces which, together with a given surface of con-
stant negative curvature, form complete center surfaces $^1$ and besides correspond to
groups of geodesic lines outgoing from the same point for the given surface, which
are differently constructed since the point is infinitely far away, or it is finite and real
or imaginary. Using Waingater’s, Beltrami’s and Dini’s theorems, Bianchi proved that
these surfaces, which correspond to the groups of parallel geodesic lines, have the same
constant curvature as the given surface, and that each surface, whose center surface is
a part of the given surface and one of these new surfaces, has the difference between
their own principal curvature radius to be constant. According to Bianchi, we can get
all those $\infty^{(1)}$ surfaces of constant curvature without integration, when for the given
surface we know one group parallel geodesic lines and their orthogonal trajectories.

But much earlier there was a theorem developed by Ribaucour, which led to the
same relation between a surface of constant negative curvature and an $\infty^{(1)}$ number
of such surfaces. Ribaucour enunciated the following theorem about deformation of
surfaces in his paper in [Comptes Rendus etc. T. LXX (1870) p. 330]: If the circles
are constructed on the tangent planes to a surface (A) and if the radii of the circles
are equal and if circles are orthogonal to $\infty^{(1)}$ surfaces, then these surfaces and the
surface (A) must be applicable at the same surface of revolution with a tractrix as
a meridian curve. It is easy to see, that those $\infty^{(1)}$ surfaces emerging from a given
surface of constant negative curvature, as trajectories orthogonal to circles, constructed
in the tangent planes of the surface, are the same surfaces of constant curvature as the
curvatures which Bianchi derived from the given surface. I illustrate this on the first
section in this paper. Nevertheless it appeared to me that Bianchi alone should have got
the credit of the remark that we can get new surfaces with the same curvature, from a
surface of constant curvature, by a certain prescribed procedure, because Ribaucour, as
far as I know, has nowhere drawn such conclusion on the theorem that I have quoted,
and this takes a secondary place on his note, there it takes place only as a special case
of a general theorem concerning to a special class of a system of $\infty^{(2)}$ circles.

Lie made an important addition to Bianchi’s theory in 1880 in [Archiv for Math-
ematik og Natyrvidenskab, 4:th Bind, Kristiania]. There he showed that we can find
those geodesic lines of new surfaces, which Bianchi derives from a surface of constant
curvature, for which the geodesic lines (or a group of parallel geodesic lines and their
orthogonal trajectories) are known, without integrating differential equation. Further-
more, we can deduce from these new surfaces an infinite number of other surfaces
of constant curvature, without integrating differential equation. And from these new
surfaces yet again an infinite number of other surfaces with the same curvature and so
one.

Short thereafter, in the same year, 1880, Lie considered more thoroughly Bianchi’s
method how to derive new surfaces of constant curvature. He proved, that in this trans-
formation principal tangent curves and curvature curves for a given surface of constant

$^1$A surface which is a locus of a center of curvature of a surface
negative curvature are transformed to surfaces with the same nature for these new surfaces, and that the principal tangent curve of the arc length will not change. \[Archiv for Mathematic etc. 5:te Bind\]. But the fact that curvature surfaces and principal tangent curves possess such invariant with respect to the previous transformation is clear from a note of Ribaucour.

Lie himself drew my attention here by me quoted work of Ribaucour's in [Comptes rendus etc. T. LXXIV (1872) p. 1399] titled "Note sur les developpees des surfaces". In this paper among other things we can see that, if for two surfaces, which together form one center surface for a surface (A), the curvature of the curves will satisfy \(^2\) to each other, then the difference between principal curvature radii for (A) must be constant, and the principal tangent curves correspond to each other, when one of those two principal curvature radii for (A) is a function of the other.

Bianchi’s theory took my attention at first by Lies work, because I founded Bianchi’s transformation such as Lie formulated, to belong to a transformation class, whose general character I have described in my paper in [Mathem. Ann. B. XVII (1880)]. This was also the reason why I paid attention to those theories of Bianchi and Lie, in the end of my paper in B. XIX. In the second section I gave a generalization of surfaces with constant curvature for this transformation, and then I worked with determining of geodesic lines of the surfaces of constant negative curvature. In my last paper in "Math. Annalen" I have observed, how we can express this problem by an equation:

\[F((x, y, dy/dx)) = dx = \text{constant}\]

there \(A, B, C\) and \(D\) are determined functions of \(x\) and contains one arbitrary parameter. After solving this equation we need only quadratures to determine geodesic lines.

## 2 Ribaucour’s and Bianchi’s theorems about surfaces of constant negative curvature.

\(\S\)1. We have a surface \(C\), a circle is constructed on the tangent plane of this surface of which the point \(p\) is the center point and \(a\) is the radius of this circle. This circle will generate a surface \(Y\) when \(p\) moves outward along a curve on \(C\). In this way these particular curves on \(C\) correspond to special surfaces \(Y\). A surface \(Y\) is an envelope of \(\infty^{(1)}\) spheres only in one case, which is tangent to that one along those generated circles. To have such envelope \(Y\), two of those consecutive generated circles \(c, c'\) must

\(\text{Two points on both surfaces correspond to each other, if they comprise the principal center of the curvature for one and the same point on (A).}\)
always belong to the same sphere, there the center of \(c\) and \(c'\) are two consecutive points \(p\) and \(p'\) on \(C\), and they are on the tangent planes of surface \(C\), and both of them have the same radius \(a\). Thus normals \(p, p'\) of the surface \(C\) must touch each other at the center of the sphere. Consequently in this case the curve which \(p\) will pass through must be a curvature curve of \(C\).

It is obvious in this case that the circles \(c, c', \ldots\) will generate the surface \(Y\) which is one of the families of curvature curves of this surface.

§2. Now I observe two different curvature curves for \(C\). They intersect orthogonally at a point \(p\). Their corresponding surfaces \(Y, Y'\) intersect along the circle \(c\), which correspond to \(p\) and which become one common curvature curve for both surfaces. The angle, where \(Y, Y'\) meet each other along \(c\), is the same angle as the angle where the spheres meet each other. The center of the spheres is two principal curvature centers of the surface \(C\) of \(p\), that goes through \(c\), since, following what we just developed, those spheres tangent to the surfaces \(Y, Y'\) along \(c\). If \(R, R'\) are two principal curvature radii of \(p\) of surface \(C\), then we get

\[
R = a \tan a, \quad R' = a \tan a',
\]

\[
a - a' = v
\]

Therefore:

\[
\tan v = a \frac{R - R'}{a^2 + RR'}
\]

We can see that \(v\) is a right angle when \(RR' = -a^2\), and we can conclude that, when \(C\) has its curvature constant \(= -1/a^2\) then surfaces \(Y, Y'\) intersect orthogonally along \(c\).

§3. Thus, When \(C\) is a surface of constant curvature equal to \(-1/a^2\), then those two families of curvature curves give two families of surfaces \(Y, Y'\), constituted in such way that every surface in one family intersects orthogonally along curvature curves all surfaces in the other families. But then, according to a theorem, there must always be a third family of surfaces, which intersects orthogonally all \(Y\) as well as all \(Y'\) along curvature curves. I denote those surfaces by \(C'\). We can define them also as surfaces intersecting orthogonally all \(\infty^{(2)}\) circles, as we showed in §1 which are constructed to all those special points on surface \(C\).

§4. \(\infty^{(2)}\) circles in §1 are orthogonal to \(\infty^{(1)}\) surfaces, only when the surface \(C\) which has the center \(p\) and their surfaces touch the point \(p\), has constant curvature equal to \(-1/a^2\). Since, as I will prove now, there is a surface, which crosses the circles \(c\) orthogonally and therefore the surface, to which the center of the circles belongs and at the same time their surfaces will touch each other, has constant curvature equal to \(-1/a^2\). Assume that the circles \(c\) on the surface \(C\) are orthogonal to a surface \(C'_0\), and presuppose that surfaces \(Y, Y'\) are constructed as two arbitrary curvature curves of \(C\), which meet each other. I denote the point where those two curvature curves
meet each other by \( p \), and its corresponding circle \( c \), which has \( p \) as center point, I denote by \( c_p \). Now, according to the assumption about the circles \( c \), the surfaces \( Y, Y' \) must be intersected by \( C_0' \) in curves, which are orthogonal to their generated circles. But according to §1 those circles are curvature curves of \( Y, Y' \). Therefore those intersection curves are also curvature curves of the same \( Y, Y' \). Once again, \( C_0' \) intersect orthogonally surfaces \( Y, Y' \) along those curves. Those curves are therefore also curvature curves for \( C_0' \). But then they are orthogonal to each other at that point, where \( C_0' \) touches the circle \( c_p \) orthogonally. Those tangent planes of \( c_p \) at this point, which touch curvature curves, are orthogonal to each other. These planes are tangent planes for \( Y, Y' \) at that point. Surfaces \( Y, Y' \) intersect along \( c_p \) under one angle, which is the same angle for each point \( c_p \). This is a right angle, as we proved above. But then, according to formula (3.2) the product of the principal curvature radius at \( p \) of surface \( C \) must be equal to \(-a^2\). Once again, those two observed curvature curves of \( C \) are two arbitrary curvature curves, which touch each other, so that \( p \) is an arbitrary point on \( C \). Thus, the curvature of the surface \( C \) must be equal to \(-1/a^2\) at each point, as we supposed before.

We summarize this on the following theorem: *If there is a surface \( C_0' \) which is orthogonal to all \( \infty(2) \) circles \( c \), and which is constructed to the points on a given surface \( C \), as in §1, then there are also other \( \infty(1) \) surfaces of the same nature as \( C_0' \), so that even they are orthogonal to the same circles \( c \), and the surface \( C \) has its curvature constant = \(-1/a^2\).*

§5. The condition for one surface \( C' \) to be orthogonal to the circles \( c \) is algebraic constructed by the equations:

\[
(x' - x)p + (y' - y)q - (z' - z) = 0,
\]

\[
(x' - x)p' + (y' - y)q' - (z' - z) = 0,
\]

\[
1 + pp' + qq' = 0,
\]

\[
(x' - x)^2 + (y' - y)^2 - (z' - z)^2 = a^2,
\]

if \( z = f(x, y) \) is the equation of the surface \( C \) then \( p = f'(x) \), \( q = f'(y) \), and if \( z' = q(x', y') \) is the equation of the surface \( C' \) then \( p' = \varphi'(x') \), \( q' = \varphi'(y') \).

¿From the symmetry system of this formula with respect to accentuated and non-accentuated letters we can see that, if \( C' \) is orthogonal to the circles \( c \), associated by §1 to the surface \( C \), then \( C \) is also orthogonal to the circles, similarly associated to the surface \( C' \). *From this and from the previous theorem follows that \( C' \) as well as \( C \) has constant curvature equal to \(-1/a^2\), and that \( C' \) in the same way yields \( \infty(1) \) surfaces \( C \), as \( C \) yields \( \infty(1) \) surfaces \( C' \).*

§6. Those three first equations on the previous § convey the condition that \( C \) and \( C' \) together form a center surface, and the fourth equation says, that the surface, whose center surface they form, has the difference between its principal curvature radius \( R \), \( R' \) of a geodesic point constant = \( a \). Thus: *every \( \infty(1) \) surface \( C' \) together with \( C \) is the center surface of a surface \( R - R' = a \).*
Surfaces $C'$, are those new surfaces which Bianchi derives from $C$ and they have constant curvature. Currently we have seen, that those surfaces form a sequence in a three dimensional orthogonal system, whose two other sequences we can obtain by algebraic operations from curvature curves of $C$. Those $C'$ are also the same surfaces that Ribaucour associated to an arbitrary surface of constant negative curvature. That they belong to an orthogonal system, according to Ribaucour, is a necessary consequence hence their orthogonal trajectories are circles.

3 Generalization of the previous theory

§7. We are searching for such kind of surfaces $z = f(x, y)$, that once again will be transmitted to surfaces $z' = \varphi(x', y')$ by the transformation, and which are decided by the equations:

$$\begin{cases}
(x' - x)p + (y' - y)q - (z' - z) = 0 \\
(x' - x)p' + (y' - y)q' - (z' - z) = 0 \\
1 + pp' + qq' - K\sqrt{1 + p^2 + q^2}\sqrt{1 + p'^2 + q^2} = 0 \\
(x' - x)^2 + (y' - y)^2 + (z' - z)^2 - a^2 = 0,
\end{cases} \tag{3.1}$$

where $K$ and $a$ are constant amplitudes

$$p = f'(x), \quad q = f'(y), \quad p' = \varphi'(x'), \quad q' = \varphi'(y').$$

In XVII band of "Mathematische Annalen $^3$" I have given a general method of how to find those surfaces in the space $r$ which will be transmitted in surfaces in the space $r'$ by a transformation which is decided by four arbitrary equations between $x, y, z, p, q, x', y', z', p', q'$. The coordinates of the points in the space $r$ are denoted by $x, y, z$, and the coordinates of the points in the space $r'$ are denoted by $x', y', z'$. $^4$ In general the correspondence between surfaces on these two possible surface groups, on $r$ and $r'$, is explicit. A surface in $r$ is corresponded by $\infty^{(1)}$ surfaces in $r'$, only when these two partial differential equations of first order will be involution, so they get $\infty^{(1)}$ common integrals, those partial differential equations we get from the transformation equation by eliminating $x, y, z, p, p'$. We denote those four transformation equations by:

$$\begin{cases}
F_1(z, x, y, p, q, z', x', y', p', q') = 0 \\
F_2(z, x, y, p, q, z', x', y', p', q') = 0 \\
F_3(z, x, y, p, q, z', x', y', p', q') = 0 \\
F_4(z, x, y, p, q, z', x', y', p', q') = 0
\end{cases} \tag{A}$$

$^3$Leipzig 1880

$^4$Both rooms $r$ and $r'$ traverse to each other, each one comprise all points in space with three dimensions.
and the involution condition is expressed by the equation:

\[(8.34)[F_1 F_2]_{x'x''} + (8.42)[F_1 F_3]_{x'x''} + (7.23)[F_1 F_4]_{x'x''} + (4.12)[F_3 F_4]_{x'x''} + (4.13)[F_1 F_2]_{x'x''} + (5.14)[F_2 F_3]_{x'x''} = 0, \quad (B)\]

\[(34), (42) etc are equations appearing subsequently: Translator's remark\]

\[
\begin{align*}
\left[ F_m F_n \right]_{x'x''} &= \frac{dF_m}{dx} \frac{dF_n}{dy} - \frac{dF_m}{dy} \frac{dF_n}{dx}, \\
\left[ F_m F_n \right]_{x'x''} &= (F_m'(x') + p' F_m'(z'))F'_n(p') + (F_m'(y') + q' F_m'(z'))F'_n(q') \\
&- (F_n'(x') + p' F_n'(z'))F'_m(p') - (F_n'(y') + q' F_n'(z'))F'_m(q').
\end{align*}
\]

\[(m, n = 1, 2, 3, 4)\]

(see Math. Annalen Bd. XVII p. 312, or the first notation at the end of this paper).

Would it now occur that even the fifth amplitude disappears by elimination of four amplitudes \(x', y', z', p', q'\), by means of (A) and (B), so that the equation (B) becomes by means of (A) an equation between only \(x, y, z, p, q, r, s, t\), then this equation will define all those surfaces in \(r\), which will be transferred to surfaces in \(r'\), and it will change those surfaces in \(r\), so that each one of these surfaces correspond to \(\infty^{(1)}\) surfaces in \(r'\). I will show that this will happen when

\[
\begin{align*}
F_1 &= (x' - x)p + (y' - y)q - (z' - z), \\
F_2 &= (x' - x)p' + (y' - y)q' - (z' - z), \\
F_3 &= 1 + pp' + qq' - K \sqrt{1 + p^2 + q^2 \sqrt{1 + p'^2 + q'^2}}, \\
F_4 &= (x' - x)^2 + (y' - y)^2 + (z' - z)^2 - \alpha^2.
\end{align*}
\]

Then we get:

\[
\left[ F_1 F_2 \right]_{x'x''} = (p - p')(x' - x) + (q - q')(y' - y) = 0
\]

(because of: \(F_1 = 0, F_2 = 0\));

\[
\left[ F_1 F_3 \right]_{x'x''} = (p - p') \left( p - Kp' \frac{\sqrt{1 + p^2 + q^2}}{\sqrt{1 + p'^2 + q'^2}} \right) \\
+ (q - q') \left( q - Kq' \frac{\sqrt{1 + p^2 + q^2}}{\sqrt{1 + p'^2 + q'^2}} \right)
= p^2 + q^2 - (pp' + qq') \left( 1 + K \frac{\sqrt{1 + p^2 + q^2}}{\sqrt{1 + p'^2 + q'^2}} \right)
+ K(p^2 + q^2) \frac{\sqrt{1 + p^2 + q^2}}{\sqrt{1 + p'^2 + q'^2}}
= (1 + p^2 + q^2)(1 - K^2)
\]
Ilir Berisha

(because of: \( F_3 = 0 \));

\[
\begin{align*}
[F_1 F_4]_{x'x'p'} &= 0; \\
[F_2 F_3]_{x'x'p'} &= 0; \\
[F_2 F_4]_{x'x'p'} &= -2(x' - x)(x' - x + p'(z' - z)) - 2(y' - y)(y' - y + q'(z' - z)) \\
&= -2a^2
\end{align*}
\]

(because of: \( F_2 = 0, F_4 = 0 \));

\[
\begin{align*}
[F_3 F_4]_{x'x'p'} &= -2 \left( p - Kp' \frac{\sqrt{1 + p'^2 + q^2}}{\sqrt{1 + p'^2 + q^2}} \right) (x' - x + p'(z' - z)) \\
&\quad - 2 \left( q - Kq' \frac{\sqrt{1 + p'^2 + q^2}}{\sqrt{1 + p'^2 + q^2}} \right) (y' - y + q'(z' - z)) \\
&= -2(z' - z) \left( 1 - K \frac{\sqrt{1 + p'^2 + q^2}}{\sqrt{1 + p'^2 + q^2}} \right) \\
&\quad - 2(z' - z) \left( pp' + qq' - K(p'^2 + q'^2) \frac{\sqrt{1 + p'^2 + q^2}}{\sqrt{1 + p'^2 + q^2}} \right) \\
&= 0
\end{align*}
\]

(because of: \( F_1 = 0, F_2 = 0, F_3 = 0 \));

Furthermore:

\[
(4.13) = \begin{vmatrix}
(x' - x)r + (y' - y)s & p'r + q's - K(pr + qs) \frac{\sqrt{1 + p'^2 + q^2}}{\sqrt{1 + p'^2 + q^2}} \\
(x' - x)s + (y' - y)t & p's + q't - K(ps + qt) \frac{\sqrt{1 + p'^2 + q^2}}{\sqrt{1 + p'^2 + q^2}}
\end{vmatrix}
\]

\[
= (rt - s^2) \left( (x' - x)q' - (y' - y)p' - K \frac{\sqrt{1 + p'^2 + q^2}}{\sqrt{1 + p'^2 + q^2}} (q(x' - x) - p(y' - y)) \right).
\]

From \( F_1 = 0 \) and \( F_2 = 0 \) it follows:

\[
(y' - y)(q' - q) = -(x' - x)(p' - p).
\]
When we substitute the value of $y' - y$ in the equation (4.13) above we obtain:

\[
(4.13) = (rt - s^2) \frac{x' - x}{q' - q} \left( q'(q' - q) + p'(p' - p) \right.
\]

\[
- K \sqrt{1 + p'^2 + q'^2} \frac{q(q' - q) + p(p' - p)}{\sqrt{1 + p^2 + q^2}}
\]

\[
= (rt - s^2) \frac{x' - x}{q' - q} \left( 1 + p'^2 + q'^2 \right) \left( 1 - K^2 \right)
\]

(because of: $F_3 = 0$).

\[
(7.24) = 2 \left| \begin{array}{c} p' - p \\ q' - q \end{array} \right| x' - x + p(z' - z)
\]

\[
= -2 \frac{x' - x}{q' - q} \left( (p' - p)^2 + (q' - p)^2 \right) + 2(z' - z)(p'q - pq')
\]

\[
= -2 \frac{x' - x}{q' - q} \left( p^2(1 + q'^2) + q'^2(1 + p'^2) + p'^2 + q'^2 \right)
\]

\[
- 2pq'q' - 2(pp' + qq')
\]

\[
= -2 \frac{x' - x}{q' - q} \left( 1 + p^2 + q^2 \right) \left( 1 + p'^2 + q'^2 \right) \left( 1 - K^2 \right)
\]

(because of: $F_1 = 0, F_3 = 0$).

The involution condition (B), which is expressed here:

\[
(8.42) [F_1 F_3]_{z'x'y'} + (4.13) [F_4 F_2]_{z'x'y'} = 0,
\]

after inserting the calculated value above for (4.13), $[F_1 F_3]_{z'x'y'}$, $[F_2 F_4]_{z'x'y'}$ when one factor $2 \frac{x' - x}{q' - q} \left( 1 + p^2 + q^2 \right) \left( 1 - K^2 \right)$ is neglected, takes this form:

\[
rt - s^2 + \frac{1 - K^2}{a^2} \left( 1 + p^2 + q^2 \right)^2 = 0.
\]

And since this equation is independent from $x', y', z'$, then it will determine, as I explained above, surfaces $z = f(x, y)$ on $r$ which we have searched for. Since the equation system (3.1) remains unchanged, when the accented letters are substituted for non-accented letters, then the surfaces $z' = \varphi(x', y')$ on $r'$, which correspond to surfaces on $r$, will also satisfy the same equation (3.2). Surfaces with constant curvature $-(1 - K^2)/a^2$ are integral surfaces of that equation.

Those surfaces are surfaces of constant curvature $-(1 - K^2)/a^2$, all those surfaces and no other surfaces.
But then it follows, according to what we have noticed at the beginning of this section that, if we have a surface of constant curvature \(-1/m^2\) and we decide \(a\) and \(K\) so that
\[
\frac{1 - K^2}{a^2} = \frac{1}{m^2},
\]
then we obtain by equations (3.1) one transformation, which will transmit the existing surface to \(\infty^{(1)}\) surfaces with the same curvature \(1/m^2\). Since the only relation between \(a\) and \(K\) is needed, which is the relation in (3.3), we get for every surface with curvature \(-1/m^2\) in all \(\infty^{(1)}\) transformations (3.1) associated. Transformation which correspond to the values \(K = 0, a = m\), is Bianchis and Ribaucours transformation given on the previous section.

¿From the form of equation system (3.1) we can see, that the points on two surfaces, which by the current transformation transform into each other, can be associated to each other pair wise in the meaning that the corresponding points become contact points of one common tangent for both surfaces, whose middle points contain segments which are constant equal to \(a\) and that the tangent planes of surfaces on both points form a constant angle equal to \(\arccos K\). Those two surfaces together form a complete center surface (§6) if the angle is right, \(K=0\).

§8. If we now have two transformations (3.1), with values \(K_1, K_2\) on \(K\), and if we let the values \(a_1, a_2\) be decided so that:
\[
\frac{1 - K_1^2}{a_1^2} = \frac{1 - K_2^2}{a_2^2} = \frac{1}{m^2},
\]
then we get two simple infinite families of surfaces with curvature \(-1/m^2\), corresponding to the same surface on \(r\) and consequently corresponding to each other. In this way we get a classification of the surfaces of constant curvature \(-1/m^2\) in reciprocal families of \(\infty^{(1)}\) surfaces. Every surface is included in two groups, each one consisting families of \(\infty^{(1)}\) surfaces.

4 Theorems about surfaces with constant curvature, derived from differential equation (3.2).

§9. In general we can have, by an arbitrary strip \(^5\), no more than one surface with a given constant curvature.

Proof. A strip is complete determined by the equations:
\[
z = f(x), \ y = \varphi(x), \ p = \psi(x), \tag{4.4}
\]

\(^5\)A strip is an infinite or a finite long but infinite thin part of a surface. The tangent plane at a point on the strip is the tangent plane of the strip.
so that the parameter $q$ of the tangent plane of the strip at the point $(x, y, z)$ satisfies equation:

$$f'(x) - \psi(x) - q\varphi'(x) = 0$$

To each element $(z, x, y, p, q)$ of the strip (4.4) we can associate a family of set of values for $r, s$ and $t$ which satisfy equations:

$$dp = rdx + sdy, \quad dq = sdx + tdy,$$

applied to the element of the strip $(z + dz, x + dx, ..., q + dq)$. Among those infinitely many sets of values there is only one set of value, which at the same time satisfies the second equation

$$rt - s^2 + \frac{1}{m^2}(1 + p^2 + q^2)^2 = 0.$$

To this set of value for $r, s, t$ we can yet again associate infinitely many sets of values for $u, v, w, \omega$, which satisfy equations:

$$dr = udx + vdy, \quad ds = vdx + wdy, \quad dt = wdx + \omega dy,$$

applied to one of the elements of the strip $(z + dz, x + dx, ..., q + dq)$ associated system of values for $r + dr, s + ds, t + dt$. When in particular the set of values for $r + dr, s + ds, t + dt$ is used, the values for $dr, ds, dt$ will satisfy the differential of the equation (3.2). Equation (2) will determine a set of values for $r, s, t$ of the element $(z + dz, x + dx, ..., q + dq)$ as it did for the element $(z, x, y, p, q)$. And the set of values for $u, v, w, \omega$, which satisfy these three equations above and one of the equations:

$$\frac{d}{dx}[rt - s^2 + \frac{1}{m^2}(1 + p^2 + q^2)^2] = 0$$

$$\frac{d}{dy}[rt - s^2 + \frac{1}{m^2}(1 + p^2 + q^2)^2] = 0$$

will satisfy even the second equation. There is no more than one set of value of this kind for $u, v, w, \omega$.

By proceeding in the same way with the forth, fifth etc. differential equations of $z$ with respect to $x, y$ we will find, by equations for the given strip and by equation (3.2) and its derivatives, that to every point on this strip is associated one and no more than one set of value for the first, the second, third etc. differential quotients of $z$.

If $x_0$, $y_0$, $z_0$ are coordinates of an arbitrary but not singular point on the strip (4.4) and if $p_0$, $q_0$, $r_0$, $s_0$, $t_0$, $u_0$, $v_0$, $w_0$, $\omega_0$ etc. are differential quotients of $z$, then we get by the series $z - z_0 = p_0(x - x_0) + q_0(y - y_0) + \frac{1}{2}(r_0(x - x_0)^2 + 2s_0(x - x_0)(y - y_0) + t_0(y - y_0)^2) + \frac{1}{2}u_0(x - x_0)^3 + 3v_0(x - x_0)^2(y - y_0) + 3w_0(x - x_0)(y - y_0)^2 + \omega_0(y - y_0)^3)$

This element is to be considered as the strip which is infinite near the point $(x, y, z)$.

I denote in this way the third partial differential equation of $z$, there $z$ is a function of $x, y$. 

---

\*\*This element is to be considered as the strip which is infinite near the point $(x, y, z)$.

\*\*I denote in this way the third partial differential equation of $z$, there $z$ is a function of $x, y$. 

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+ etc.

a surface for the convergence area of this sequence, which contains the strip (4.4) and which is an integral of (3.2), hereby the theorem is proved.

We can express this in following way: There is one and only one surface with a given constant curvature, which touches a given surface along a given curve.

§10. Equations:

\[ dp = r dx + s dy, \quad dq = s dx + t dy, \quad rt - s^2 + \frac{1}{m^2} (1 + p^2 + q^2)^2 = 0, \quad (4.5) \]

\[ (dy = \varphi'(x) dx, \quad dp = \psi'(x) dx) \]

which determine the set of values for \( r, s, t, \) which are associated to the element \((z, x, y, p, q)\) of the integral of (3.2) and which should pass through the strip (4.4), leaves for \( t \) this value:

\[ t = \frac{dq^2 - \frac{1}{m^2} (1 + p^2 + q^2)^2 dx^2}{dpdx + dqdy} \quad (4.6) \]

consequently \( t \) become unknown, if at the same time:

\[ dpdx + dqdy = 0 \quad (4.7) \]

and

\[ dq^2 = \frac{1}{m^2} (1 + p^2 + q^2)^2 dx^2, \quad (4.8) \]

on the other hand \( t \) become infinite large, if equation (4.7) is satisfied, but not equation (4.8). In this latter case the strip (4.4) become one singular strip, a cuspidal strip for the integral of (3.2), that could pass through the same one. Nevertheless the proof on the previous § will lose its validity, so we leave it undecided whether it can always be associated a surface with the curvature \(-1/m^2\), by an arbitrary strip of the property as (4.8). In the first case, when for the strip (4.4), as well (4.7) as

\[ dq = \frac{1}{m} (1 + p^2 + q^2) dx \quad (4.8') \]

or

\[ dq = -\frac{1}{m} (1 + p^2 + q^2) dx, \quad (4.8'') \]

all \( \infty^{(1)} \) set of values for \( r, s, t, \) will be determined, and even equation (3.2) will be satisfied. We find now \( \infty^{(1)} \) set of values for \( u, v, w, \omega, \) which become by the previous strip (4.4) associated to the element \((z, x, y, p, q)\) and one of the \( \infty^{(1)} \) set of values for \( r, s, t, \) and which satisfies the first derivative of the equation (3.2). According to the
first proof, we can now draw through the strip $\infty^{(\infty)}$ number integral surfaces of (3.2) an infinite number of integral surfaces, which has along the strip a contact of the first order, and an infinite number of integral surfaces with a contact of the second order, etc.

§11. Equation (4.7) expresses the strip (4.4), which joins to the curve: $z = f(x)$, $y = \varphi(x)$, so its tangent planes at the point $(x, y, z)$, $(x + dx, y + dy, z + dz)$ intersect on the tangent to the curve at the point $(x, y, z)$. In other words, so that tangent planes of the strip are osculating planes of the curve. Equation (4.8) says, that the torsion radius of the curve is constant $= m$\textsuperscript{8}, hereby our statement is proved.

For each one of these $\infty^{(\infty)}$ surfaces with curvature $-1/m^2$, which goes along the current strip, the curve becomes one principal tangent curve, $^9$ to which the strip joins.

The strip on a surface of constant curvature $-1/m^2$, which touches some principal tangent curve to the surface has the property expressed by the equation (4.7). Furthermore equation (4.6) must leave at least one finite value for $t$, which is the value of $(d^2z)/(d^2y)$, and which is obtained from the equation of the surface. Thus, even the equation (4.8) must be satisfied by the strip. Hence it follows, that every surface of constant curvature is touched, along each of its principal tangent curves, by $\infty^{(\infty)}$ other surfaces with the same curvature. And principal tangent curves to a surface of constant curvature $-1/m^2$ has constant torsion radii $= 0$.

Contact strips between integral surfaces of a partial differential equation are denoted as the characteristics of the equation. After what is developed here above, the characteristics of the differential equation (3.2) are principal tangent curves of its integral surfaces. And every curve of constant torsion radii $= m$ is a characteristic to the equation (3.2).

§12. If we have a surface with a given constant curvature, which is determined by the condition of having one given strip, e.g. by the condition of having one given curve as curvature curve and at the same time at one given point on that curve touches one given tangent plane. On the other hand, a surface with a given constant curvature can not be determined so that it gets an arbitrary curve as principal tangent curve. To possess this importance for a surface of constant curvature $-1/m^2$, the curve must have the horizontal radius constant $= m$. And with such kind of curve there are $\infty^{(\infty)}$ surfaces of constant curvature for which all curves have the same function as principal tangent curve. The curvature of the surfaces is now necessary $= -1/m^2$.

"Enneper" has remarked that the principal tangent curves have their torsion radii $= m$. He has also showed that, for any surface the sum of the square of the torsion radii of a principal tangent curve at an arbitrary point and the inverse value of curvature of

\textsuperscript{8}Because if we choose the tangent of the curve at the point $(x, y, z)$ on X-axis and if we choose its osculated plane at the same point on XY plane, then we have $p = q = 0$ and $dq = d\cos(ZY) = -\sin(ZY)d(ZY) = -d(ZY) = \text{osculated angle } d\sigma$. And equation (8) now gets the form: $(\frac{d\sigma}{ds})^2 = 1/m^2$

\textsuperscript{9}This is a curve whose tangents touches the surface at three points.
the surface at the same point will disappear. (Göttingische Nachrichten fr 1870, p. 499. Mathematische Annalen B.II, p. 596).

§13. What is the condition that two infinite nearby characteristics will belong to the same surface?

We have two consecutive elements \((z, x, y, p, q)\), \((z + dz, ... , q + dq)\) of a characteristic:

\[
dy = y'dx, \quad dz = (p + qy')dx, \quad dp = -\frac{1}{m}(1 + p^2 + q^2)y'dx, \quad dq = \frac{1}{m}(1 + p^2 + q^2)dx. \tag{4.9}
\]

\(z + \delta z, x + \delta x, ..., q + \delta q\) are parameters of an element to an infinite nearby characteristic, and \(z + dz + \delta z + d\delta z ... q + dq + \delta q + d\delta q\) or \(z + \delta z + dz + d\delta z ... q + \delta q + dq + d\delta q\) are parameters of the next element to the same characteristic. The transition from the first characteristic to the second one is denoted by \(\delta\), and by \(d\) is denoted an infinitely small displacement towards the first characteristic. We have operation \(d\delta\) equal to the operation \(\delta d\). The condition that those four elements will belong to the same surface is given by the equation:

\[
\delta z = p\delta x + q\delta y \tag{4.10}
\]

and by one equation which is obtained by its differentiation.

From the second part of the equation (4.9) it follows:

\[
\delta dz = (\delta p + y'\delta q)dx + (p + qy')\delta dx + qdx\delta y',
\]

and from (4.10), considering last two parts of the equation (4.9):

\[
d\delta z = p\delta dx + q\delta dy + dx \frac{1 + p^2 + q^2}{m} (\delta y - y'\delta x).
\]

Since \(dy = y'dx\), we get: \(\delta dy = \delta dy = y'\delta dx + dx\delta y'\) and therefore:

\[
d\delta z = (p + qy')\delta dx + dx \left[ \frac{1 + p^2 + q^2}{m} (\delta y - y'\delta x) + y\delta y' \right],
\]

hence \(d\delta z = \delta dz : \delta p + y'\delta q = \frac{1}{m}(1 + p^2 + q^2)(\delta y - y'\delta x). \tag{4.11}\)

We will use this equation to determine \(\delta y'\).

Those two elements \((z, x, y, p, q)\), \((z + \delta z, x + dx, ..., q + dq)\) are consecutive elements to a characteristic which touches these two assumed elements. There must
bee a similarity, if those strips shall belong to the same surface and to the same integral surface of the equation (3.2), and equation (4.11) must be satisfied by such an characteristic associated to the values of $\delta p, \delta q$:

$$\delta p = \frac{1}{m}(1 + p^2 + q^2)\delta y, \quad \delta q = -\frac{1}{m}(1 + p^2 + q^2)\delta x. \quad (4.12)$$

By differentiating (4.11) we obtain:

$$\frac{d}{dx}(\delta p) + y'\frac{d}{dx}(\delta q) = 2 \frac{m}{m}(p\frac{dp}{dx} + q\frac{dq}{dx})(\delta y - y'\delta x)$$

$$+ \frac{1}{m}(1 + p^2 + q^2)(\frac{d}{dx}(\delta y) - y'\frac{d}{dx}(\delta x) - \frac{dy'}{dx}\delta x).$$

or, when

$$\frac{d}{dx}(\delta f) = \frac{\delta df}{dx} = \frac{\delta f \delta dx}{dx \ dx}$$

or according to (4.9)

$$p\frac{dp}{dx} + q\frac{dq}{dx} = \frac{1}{m}(1 + p^2 + q^2)(q - py') :$$

$$\delta(\frac{dp}{dx}) + y'\delta(\frac{dq}{dx}) = \frac{dp}{dx} \frac{dx}{dx} + y'\frac{dq}{dx} \frac{dx}{dx} + \frac{dy'}{dx} \delta q$$

$$= \frac{2}{m^2}(q - py')(1 + p^2 + q^2)(\delta y - y'\delta x)$$

$$+ \frac{1}{m}(1 + p^2 + q^2)(\delta y' - \frac{dy'}{dx}\delta x),$$

or, since $dp + y'dq = 0$ and according to (4.12)

$$\delta(\frac{dp}{dx}) + y'\delta(\frac{dq}{dx}) = \frac{2}{m^2}(q - py')(1 + p^2 + q^2)(\delta y - y'\delta x)$$

$$+ \frac{1}{m}(1 + p^2 + q^2)\delta y'.$$

But according to (4.9):

$$\delta(\frac{dp}{dx}) = -\frac{1}{m}(1 + p^2 + q^2)\delta y' - \frac{2}{m}(p\delta p + q\delta q)y',$$

$$\delta(\frac{dq}{dx}) = \frac{2}{m}(p\delta p + q\delta q),$$

thus:

$$\delta(\frac{dp}{dx}) + y'\delta(\frac{dq}{dx}) = -\frac{1}{m}(1 + p^2 + q^2)\delta y',$$
finally, by comparing these two obtained expressions for $\delta \left( \frac{dp}{dx} + y' \delta \left( \frac{dq}{dx} \right) \right)$ we get:

$$\delta y' + \frac{1}{m} (q - py') (\delta y - y' \delta x) = 0. \quad (4.13)$$

Now we denote the characteristic which was mentioned here above by $h$, and the elements $(z, x, y, p, q)$, $(z + \delta z, x + \delta x, \ldots, q + \delta q)$ are on this characteristic. Its equations are:

$$y = \varphi(x), \quad p = p(x), \quad q = q(x)$$

Equation (4.13) transforms now into:

$$m \frac{\delta y'}{\delta x} + py' - (q + p \varphi'(x)) y' + q \varphi(x) = 0. \quad (4.13')$$

The curve to which the strip $h$ joins, according to the developments on this section, becomes a principal tangent curve for those integral surfaces of (3.2) which contain the strip. We observe one of these surfaces. $P$ is an arbitrary point $(x_0, y_0, z_0)$ on $h$. One of the principal tangents of the surface at the point $P$ touches the curve, to which $h$ joins. We denote the second principal tangent at the point $P$ with $PT$. And we denote the value of $y'$ by $y'_0$ and its direction by $PT$. There is a solution $y' = F(X)$ of the equation (4.13), which is constituted on the way that we get $F(X_0) = y'_0$ for $P$. By this value, $F(x)$, on $y'$ the principal tangent to the surface at an arbitrary point on $h$, becomes complete determined.

If we choose the tangent plane of the strip at the point $(x, y, z)$ to the plane $xy$, then we get $p = 0, q = 0$ and thus from (13') $\delta y' = 0$. It means that, if $P, P', P'', \ldots$ are consecutive points on $h$ and if $PT, P'T', P''T'', \ldots$ are the other principal tangents than $PP', P'P'', \ldots$ for the previous integral surface of (3.2), which goes through $h$, then we get $P'T'$ by drawing through $P'$ one parallel to $PT$ and then we rotate it around $PP''$ until it falls on the tangent plane of the strip $h$ at $P'$. In similar way we obtain $P''T'', \ldots$, etc. If we have now new principal tangent curve of that surface, which goes through $P$ and which has the tangent $PT$ and a point $T$ infinite near $P$, then we get the second tangent to this surface on $T$ which touches the last principal tangent curve, by drawing through $T$ a parallel to $PP'$ and rotate it around $PT$ as we rotate an axle, until it falls on the osculated plane of the curve at $T$.

If $k$ is the last principal tangent curvature, then it is clear that the point $P'$ describes one principal tangent curve of the integral surface which is infinite nearby $k$, this by giving successive $\infty^{(1)}$ translations outwards $k$ and $\infty^{(1)}$ rotations around $k$. Those translations and rotations are connected with each other pair wise. Hereby the answer is given for the posed question at the beginning of this paper.

We can summarize this in two theorems:

Through two curves of constant torsion radii $= 0$, which touch each other at a point and which are osculated by one single plane, goes one determined surface of constant curvature $-1/m^2$. 
Every surface of constant curvature can be divided into $\infty^{(2)}$ equilateral quadrangles by its principal tangent surfaces.

5 Determination of surfaces that are obtained by a given surface of constant curvature $-1/m^2$, from transformation (3.1) in section 3.

§14. Every strip on $r$:

$$z = f(x), \quad y = \varphi(x), \quad p = \psi(x), \quad q = \frac{f'(x) - \psi(x)}{\varphi'(x)}, \quad (5.14)$$

is transmitted by the transformation (3.1) into $\infty^{(1)}$ strips $r'$. These strips are integrals of those three differential equations of the first order, which are emerged by eliminating $x, y, z, p, q$ between equation (3.1) and (5.14). But we can define those also geometrically as on equation (3.1).

On the given tangent plane of the strip, we construct a circle at the point $p$ with center $p$ and radius $a$. This circle describes a surface $Y$, when $p$ moves outwards the strip. If $\pi$ is an arbitrary point on the surface $Y$ and if $p$ is the contact point of that given strip with the plane $Y$ which goes through $\pi$, and if we construct a plane, which goes through $\pi p$ and cross the circle plane under the determined angle $\theta = \arccos K$, and then if we calculate the angle $\theta$ from the circle plane on the direction of the rotation which was predetermined, then the point $\pi'$ which falls infinite near $\pi$ on the new intersection with $Y$, together with $\pi$ will belong to one of $\infty^{(2)}$ strips on $r$. The plane becomes the tangent plane of the strip at $\pi$. By means of the point $\pi'$ and the point $p$ on the strip (5.14) whose tangent plane for this strip goes through $\pi'$, we construct in the same way a new point, $\pi''$, on the same strip which contain $\pi$ and $\pi'$ and which is one of these $\infty^{(1)}$ strips, etc. All those $\infty^{(1)}$ surfaces to which those $\infty^{(1)}$ strips join on $r'$ are on $Y$ and their points as well as their tangent planes are successive constructed, as we have shown above. This construction of the strips on $r'$, that correspond to an arbitrary strip on $r$, leads to a simple analytic expression of those curves to which this strip joins.

$p$ and $p'$ are two infinite nearby points on the given strip, as we showed before. They are the center of two circles on the tangent plane of the strip, both with a radii $= a$. I denote these circles by $c, c'$. $\pi$ is an arbitrary point on $c$, and $\pi'$ on $c'$ and it belongs to one of the sought strips which goes through $\pi$, so the plane $p\pi\pi'$ becomes the tangent plane of this strip at $\pi$.

Furthermore $p'T'$ characterizes the line of intersection between planes of the circles $c, c'$, which is the given conjugate tangent to $pp'$. $pT$ is the conjugate tangent to the $pp'$. $pT'$ is conjugate tangent to $pp'$ for every surface that contains the given strip.
nearest previous line element of the curve, to which the strip join.

If we trace from $\pi'$ a normal to the plane $Tp\pi$ or $T'p'\pi'$ and if we denote by $\pi'$ the point where this normal touches that plane, then we get:

$$T'p'\pi' = Tp\pi + dr' - p\pi p' + \pi p'\pi', \quad (5.15)$$

if the angle between $pT$ and $P'T'$ is denoted by $dr'$.

Here we put

$$Tp\pi = \omega, \quad T'p'\pi' = \omega + d\omega, \quad (5.16)$$

Therefore we obtain also the angle between planes $T'p'\pi'$ and $T'p'\pi'$, which is the tangent plane of the strip at $p$ and $p'$, and which is infinitely small, I denote this angle by $d\sigma$,

$$T'p'\pi' = \omega + d\omega. \quad (5.17)$$

We can easily see that the angle $pp'$ satisfies equation:

$$pp' \sin pp'\pi = a \sin P\pi p',$$

or, when $pp' = ds$, $Tp\pi = 180^0 - \alpha$:

$$pp' = ds \sin (\omega - \alpha). \quad (5.18)$$

The angle $\pi'p'\pi'$ is a projection on the plane $T'p'\pi$, which is the tangent plane of the strip at $p$, of angle $\pi p'\pi'$, whose plane form an angle $\theta$ with the previous plane, when we write:

$$K = \cos \theta, \quad (5.19)$$

thus:

$$\pi'p'\pi' = \pi p'\pi' \cos \theta.$$

Furthermore we can see, if we consider spherical triangle on the sphere with center $p'$ which has its angle on the direction of the lines for $p'T', p'\pi', p'\pi'$, that

$$\frac{\sin \pi'p'\pi'}{\sin T'p'\pi'} = \frac{d\sigma}{\sin \theta'},$$

\textsuperscript{11} + an infinite small amplitude of the same order as $ds$
there the difference between $\theta'$ and $\theta$ is infinitely small, the difference between $\pi p'/p'$ and $\omega$ is infinitely small to. Therefore:

$$\pi p'/p' = d\sigma \frac{\sin \omega}{\sin \theta},$$

and thus:

$$\pi p'/p' = d\sigma \cot \theta \sin \sigma. \quad (5.20)$$

By substituting (5.16), (5.17), (5.18), (5.20) formula (5.15) becomes:

$$d\omega = dr' - \frac{ds}{a} \sin(\omega - \alpha) + d\sigma \cot \theta \sin \omega$$

or

$$\frac{d\omega}{ds} = \frac{dr'}{ds} + \frac{d\sigma}{ds} \cot \theta \sin \omega - \frac{1}{a} \sin(\omega - \alpha). \quad (5.21)$$

Here $\alpha$, $dr'/ds$, $d\sigma/ds$ are some of the properties of the strip, of its equations, determined functions of $s$.

By the integral of the equation (5.21): $\omega = F(s, C)$ there $C$ is an arbitrary constant, the curves on $Y$ will be determined, to which the $\infty(1)$ strips on $r'$ join, which correspond to the given strip on $r$. $\sigma, s$ are some kind of Gauss coordinates of the points on $Y$. One of these generated circles $c, c', \ldots$ of the surface is in particular represented by the equation $s = \text{constant}$.

§15. Equation (5.21) can be written in this way:

$$\frac{d\omega}{ds} = A \sin \omega + B \cos \omega + C,$$

where

$$A = \cot g \frac{d\sigma}{ds} - \frac{1}{a} \cos \alpha,$$

$$B = \frac{1}{a} \sin \alpha,$$

$$C = \frac{d\sigma}{ds}.$$

If

$$e^\omega \sqrt{-1} = z,$$

and

$$\sin \omega = \frac{z^2 - 1}{2z\sqrt{-1}}, \cos \omega = \frac{z^2 + 1}{2z}.$$
then we obtain equation for the sought strips on $r'$:

\[ \frac{dz}{ds} = \frac{1}{2}(A + B\sqrt{-1})z^2 + C\sqrt{-1}z - \frac{1}{2}(A - B\sqrt{-1}) \tag{5.21'} \]

This equation is of Riccati type. If we know a particular integral to this equation, then we can get the general solution of the equation using only quadratures. Therefore:

If one of those strips on $r'$, which correspond to a given strip on $r$, is known, then other infinite strips are obtained using only quadratures.

§16. When $A + B\sqrt{-1} = 0$ or $A - B\sqrt{-1} = 0$, then equation (21') becomes linear of this form:

\[ \frac{dy}{ds} + Py + Q = 0, \]

there $P, Q$ are determined functions of $s$, and can be integrated using only quadratures. Consequently those infinite strips on $r'$, which correspond to a strip on $r$, are determined using only quadratures, for which:

\[ a \cot \theta \, d\sigma - ds \, e^{\pm\alpha\sqrt{-1}} = 0 \]

When $\theta = 90^0$ then this theorem gets the following expression: Those infinite strips on $r$, which are emerged from Bianchi's transformation of a strip on $r$, which joins to a curve with null length*, are described by the equation:

\[ \omega = \tau' + \text{one arbitrary constant}. \]

§17. We can see that

\[ \pi\pi'^2 = \pi\pi'^2 + \pi'\pi'^2. \]

Furthermore:

\[ \pi\pi'^2 = a^2 \sin^2 \pi'\pi' + \pi'\pi'^2 \cos^2 \pi'\pi' \]

or, according to (5.20):

\[ \pi\pi'^2 = a^2 d\sigma^2 \cot^2 \theta \sin^2 \omega + ds^2 \cos^2 (\omega - \alpha). \]

And, since we go from $\pi'$ to $\pi'_1$ by a rotation of the plane $T'p'\pi'$ around $T'p'$ one angle $d\sigma$, so that it coincides with the plane $T'p'\pi$, where $\pi'$ describes an arc element of a circle with radius $a \sin \omega$, then it follows:

\[ \pi'\pi'^2 = a^2 \sin^2 \omega d\sigma^2. \]

Therefore, if instead of $\pi'\pi'$ we write $ds'$:

\[ ds'^2 = a^2 \sin^2 \frac{\omega}{\sin^2 \theta} d\sigma^2 + \cos^2 (\omega - \alpha) ds^2, \]

\(^{12}\text{Even equation (13') is of this type.}\)
or when

\[ m^2 = \frac{a^2}{\sin^2 \theta} \]  

(3.3')

\[ ds'^2 = m^2 \sin^2 \omega d\sigma^2 + \cos^2(\omega - \alpha)ds^2. \]  

(5.22)

§18. For a strip on \( r \), which is of the nature that it can belong to a surface of constant curvature \(-1/m^2\) and joins to a principal tangent curvature of this surface, then according to the development in §11, we have:

\[ \alpha = 0, \quad m d\sigma = ds \]

Consequently, according to the previous formula (5.22), for this case:\(^{13}\)

\[ ds' = ds. \]

§19. In §7 is proved, that the transformation (3.1) will transmit every surface with curvature \(-1/m^2\) into \(\infty^{(1)}\) other similar surfaces. The same transformation will transmit every element \((z, x, y, p, q)\) into \(\infty^{(1)}\) other similar elements \((z', x', y', p', q')\). Consequently every contact strip between two surfaces with the curvature \(-1/m^2\) will transmit into \(\infty^{(1)}\) other strips, each one is a contact strip between two other surfaces with the same curvature \(-1/m^2\). With other words: every characteristic (§11) of the partial differential equation (3.2) is transmitted into \(\infty^{(1)}\) other characteristics.

But the characteristics of (3.2) are (§11) strips on the surfaces with the curvature \(-1/m^2\) along their principal tangent curves; thus every principal tangent curve of a surface with curvature \(-1/m^2\) will be transmitted by the transformation (3.1) into \(\infty^{(1)}\) principal tangent curves for the same number other surfaces with the curvature \(-1/m^2\).

If \( C' \) is one of \(\infty^{(1)}\) surfaces on \( r' \), which correspond to a given surface \( C \) with curvature \(-1/m^2\), and if the surface \( C \) is divided by its principal tangent curve into \(\infty^{(1)}\) equilateral quadrilaterals (§13), then according to the formula on the previous §, this net of quadrilaterals will be transmitted into an other similar net on \( C' \), and sides of a quadrilateral on \( C' \) are equal to the sides of one of quadrilaterals on \( C \).

These quadrilaterals on \( C \) and \( C' \) correspond to each other one to one, so that every corner on a quadrilateral on \( C \) correspond to a particular corner on a particular quadrilateral on \( C' \). One diagonal to the previous quadrilateral corresponds one diagonal to the latter one. But the diagonals become pieces of curvature curves of \( C \) and \( C' \), as diagonals and curvature curves halve angels between principal tangent curves of \( C \) and \( C' \). Therefore:

Not only principal tangent curves of \( C \), but also curvature curves of \( C \), are transformed by (3.1) into curves of the same nature for \( C' \).

---

\(^{13}\) this is a curve whose tangents touch the circle which is infinite far away
§20. Surface $Y$ which correspond to a principal tangent curve of $C$ (§1), according to the first theorem on the previous §, will be touched by $\infty^{(1)}$ surfaces $C'$, which correspond to $C$, on principal tangent curves for the same $C'$. Those curves are divided in pieces, of the same length, by circles which generate surface $Y$ (as in §18).

If $K = 0$ then those curves are orthogonal to those circles that generate surface $Y$, and their osculating planes are also orthogonal to the same circles, and consequently go through the normal to the surface $Y$. But then they become geodesic lines of $Y$. Hence it follows that their orthogonal curves divide circles which generate $Y$ into two equal pieces.

§21. After developing §14 it is easy to derive surfaces $C'$ when $C$ is given. We divide $C$ arbitrary into $\infty^{(1)}$ strips $S, S', S'', \ldots$ lying next to each other and then we draw a strip $\sum$ which intersect all those surfaces. Surfaces $Y, Y', Y'', \ldots$ which correspond to those curves and to which $S, S', S'', \ldots$ joins are constructed in this way. The surface of the strip $\sum$ is constructed in the same way and is of the same nature. I call this surface $U$. By equation (5.21) we seek those curves on $Y, Y', Y'', \ldots U$ to which the the strips on $r'$ join, and which correspond to $S, S', S'', \ldots \sum$. Those curves are intersection curves to $Y, Y', Y'', \ldots U$ of the surfaces which we are searching for, because, according to what we have shown in §7, the strips on $r'$ will join to $\infty^{(1)}$ surfaces, to surfaces $C'$.

We get a special surface from surfaces $C'$ by selecting, among those found curves on the surfaces $Y, Y', Y'', \ldots$ those curves that meet the same curves on $U$.

But we can choose strips $S, S', S'', \ldots$ so they go to the same point or they have one common element $(z_0, x_0, y_0, p_0, q_0)$. In this case we can let $\sum$ be of this common element, and of the determined curves of the equation (5.21) on $(x_0, y_0, z_0)$ and generates one of the surfaces $C'$.

We obtain surfaces $C'$ by integrating equation (5.21) with an arbitrary parameter.

§22. It is clear from §15 that if we know one of the surfaces $C'$, we get other surfaces using only quadratures. Of those $\infty^{(1)}$ surfaces on $r$ which correspond to one of these latter surfaces, one surface is known, namely surface $C$, and we have by quadratures other surfaces on $r$, which correspond to them. Of the same reason we can get from the last $\infty^{(2)}$ surfaces their $\infty^{(3)}$ corresponding surfaces on $r$, using only quadratures etc. Thus: If two surfaces with curvature $-1/m^2$ are given, on $r$ and $r'$, and if they correspond to each other as (3.1), then we have, using only quadratures $\infty^{(\infty)}$, other surfaces with the same curvature $-1/m^2$.

6 The geodesic lines of surfaces of constant negative curvature

§23. LEMMA I. Those surfaces, for which two given surfaces form the complete
center surface, are obtained using only quadratures.

This theorem is known since before and we can prove it as follows: The surfaces we are looking for are orthogonal to the common tangents of the given surfaces. If \( x = rz + \varrho, \ y = sz + \sigma \) are equations of a common tangent, then \( r, s, \varrho, \sigma \) must satisfy two equations:

\[
F(r, s, \varrho, \sigma) = 0, \\
\Phi(r, s, \varrho, \sigma) = 0,
\]

which we get, by algebraic operations, from given equations of the surfaces. Furthermore, if

\[
f(x, y, z) = C
\]

is the equation of surfaces we are looking for, then

\[
r = \frac{f'(x)}{f'(z)}, \ s = \frac{f'(y)}{f'(z)}, \ \varrho = \frac{xf'(z) - zf'(x)}{f'(z)}, \ \sigma = \frac{yf'(z) - zf'(y)}{f'(z)}.
\]

By substituting those expressions of \( r, s, \varrho, \sigma \) on \((a)\) we obtain:

\[
F\left(\frac{f'(x)}{f'(z)}, \frac{f'(y)}{f'(z)}, \frac{xf'(z) - zf'(x)}{f'(z)}, \frac{yf'(z) - zf'(y)}{f'(z)}\right) = 0, \\
\Phi\left(\frac{f'(x)}{f'(z)}, \frac{f'(y)}{f'(z)}, \frac{xf'(z) - zf'(x)}{f'(z)}, \frac{yf'(z) - zf'(y)}{f'(z)}\right) = 0.
\]

two partial differential equations of the first order, which have one common solution: \( f = (x, y, z) \). But it is also clear that surfaces \((b)\) are parallel surfaces so the equation \( f(x', y', z') = C^0 + dC \) of a surface which is infinite near the surface \( f(x, y, z) = C^0 \) can also be written:

\[
f(x, y, z) + f'(x)\delta x + f'(y)\delta y + f'(z)\delta z = C^0 + dC,
\]

if

\[
\delta x = \varepsilon \frac{f'(x)}{\sqrt{f'(x)^2 + f'(y)^2 + f'(z)^2}}, \\
\delta y = \varepsilon \frac{f'(y)}{\sqrt{f'(x)^2 + f'(y)^2 + f'(z)^2}}, \\
\delta z = \varepsilon \frac{f'(z)}{\sqrt{f'(x)^2 + f'(y)^2 + f'(z)^2}}.
\]
Hence it follows that the equation,

\[ f'(x)^2 + f'(y)^2 + f'(z)^2 = 1 \]

must be compatible with (c), if we put \( f \) instead of \((\varepsilon/dC)f\). But from (c) and (d) we obtain, by elimination:

\[ f'(x) = A(x, y, z), \quad f'(y) = B(x, y, z), \quad f'(z) = C(x, y, z), \]

and therefore we need only performance of the integration:

\[ \int (Adx + Bdy + Cdz) \]

so that equation \((b)\) will appear.

(Lemma II). *The characteristics of the partial differential equation of the first order, which determine surfaces, for which one a given surface is a part of the center surfaces, become curvature curves of the previous surfaces. Cuspidal curve of the developing surface is a geodesic line for that given surface. This developing surface is formed from normals of the same surfaces at the points on one of these curvature curves.*

This theorem is a part of Lie’s theorem concerning integrals of a general ball-complex. Those surfaces are integrals of a ball-complex, made of those balls, whose center is on that given surface. And according to Lie the characteristics of a ball-complex are curvature curves of integral of the complex. But for this special theorem the proof can be performed in the following way.

Surfaces which have a given surface \(C\) as a part of their center surfaces, can also be defined as surfaces whose normals touch the given surface \(C\). Those surfaces are expressed by an equation of the type \((c)\). Characteristics for this equation are intersect curves between two infinite nearby integral surfaces of the equation. Two infinite nearby surfaces \(\Gamma, \Gamma'\), intersect on a curve at an arbitrary point \(p\), the tangent to the curve \(pp'\) is orthogonal to the plane, which contain normals of the surfaces at the point \(p\). This plane touches \(C\), when \(C\) is a part of the center surfaces of \(\Gamma\) and \(\Gamma'\). But the same one is also a tangent plane at \(p\) of a curvature curve for \(\Gamma\). Hence it follows that, \(pp'\) is a tangent to the second curvature curve of \(\Gamma\), which goes through \(p\), and the normal of this surface at \(p\) and at the border of next point of the last curvature curves touch each other at contact point of the surface \(C\) with the previous plane. I denote this contact point \(\pi\). Orthogonal to this plane is that one which contains two latter normals. But this last plane become an osculating plane for the curve on \(C\), which \(\pi\) describes, when \(p\) moves outward intersection curve between \(\Gamma\) and \(\Gamma'\), which is a curvature curve of \(\Gamma\). Therefore the osculating plane of the locus of \(\pi\) at an arbitrary point on locus goes through the normal of the surface \(C\) at the same point, then the previous locus of \(\pi\) must be a geodesic line of \(C\). As we claimed here above.
§24. We have proved in §6, that each of $\infty^{(1)}$ surfaces $C'$, which by Bianchi’s transformation (3.1) for $K = 0$, is derived by a given surface $C$, whose curvature is constant $= -1/a^2$, form with $c$ the complete center surface for a surface, to which $R - R' = a$. Parallel surfaces to a similar surface have the same property: $R - R' = a$. When surfaces $C'$ are found, surfaces $R - R' = a$ are obtained using only quadratures according to (lemma I), which together with $C'$ are center surfaces. Those surfaces, which form $\infty^{(1)}$ groups of $\infty^{(1)}$ parallel surfaces between themselves, are a complete solution to the partial differential equation of the first order, which define all surfaces, to which $C$ is a part of the center surfaces. Characteristics of this differential equation are obtained by differentiation of the equation for surfaces $\infty^{(2)}$: $R - R' = a$ with respect to two arbitrary parameters in the equation. This because they are intersection curves between pair wise infinite nearby integral surfaces. But the characteristics are, according to (lemma II) curvature curves for surfaces $R - R' = a$. From curvature curve of a surface we get once again, by algebraic operations, differentiations and eliminations, one geodesic line for the center surface of the surface, a line which has curvature curve as evolvent. According to the second part of lemma II characteristics must follow $\infty^{(2)}$ geodesic curves of $C$.

Thus: Geodesic lines of the surface of constant curvature $1/a^2$, are decided, by a Riccati’s equation (5.21') with an arbitrary parameter, ($\theta = 90^0$) and by quadratures.

In §22 has been shown that, if one of the surfaces is known, other surfaces can be obtained using only quadratures, consequently:

If we have a surface of constant negative curvature, and if we know another surface with the same curvature, which together with the first one form a complete center surface, than we can obtain using only quadratures geodesic lines for both surfaces $^*$. From the last theorem in §16 we can conclude, that if surfaces of constant negative curvature with null length are known, then their geodesic lines can be obtained using only quadratures.

This appears even on ”Liouville’s theorem of line element of a curve on a surface of constant curvature, when curves of the null length on surfaces are used as coordinate system. (See. Lie: Zur Theorie der Flichen konstanter Krmmung. Archiv for Mathematik og Naturvidenskab. Fjrde Bind. P.363-366)

7 Consequences of formula (5.22) in §4

§25. If we want to use formula (5.22) to decide the line element on $C'$, which by transformation (3.1) corresponds to one line element $ds_1$ of a curvature curve for $C$, then we put $\alpha = 90^0$, $d\sigma = ds_1/R_1$ when $R_1$ is curvature radius of the normal section through $ds_1$ at the point $p$, which is the point, from which $ds_1$ starts. If we denote the
searching line element by $ds'_1$, then we get:

$$ds'_1^2 = ds_1^2 \sin^2 \omega \left( 1 + \frac{m^2}{R_1^2} \right)$$

or there $m^2 = -R_1 R_2$, if $R_2$ means curvature radius at $p$ of normal section orthogonal to $ds_1$:

$$ds'_1 = ds_1 \sin \omega \sqrt{\frac{R_1 - R_2}{R_1}},$$

or

$$ds'_1 = ds_1 \cos \omega \sqrt{\frac{R_1 - R_2}{R_1}}, \quad (7.23)$$

if $\omega_0$ is the angle between the direction of $ds_1$ and $p\pi$, there $\pi$ characterize the point on $C'$, which correspond to $p$.

According to §19, $ds_1$ is an element of curvature curve of $C'$. If $ds_2$ is an element at $p$ of curvature curve of $C$, which is orthogonal to $ds_1$, then we get the corresponding line element on $C'$, which we denote by $ds'_2$, by writing: $\alpha = 90^0$, $\omega = \omega_0$, $d\sigma = ds_2/R_2$, in equation (5.22) therefore we get:

$$ds'_2 = ds_2 \sin \omega_0 \sqrt{\frac{R_1 - R_2}{R_2}}. \quad (7.24)$$

Hence it follows that this new expression of the line element $ds' = \sqrt{ds'_1^2 + ds'_2^2}$ which correspond to the line element $ds = \sqrt{ds_1^2 + ds_2^2}$:

$$ds'^2 = \left( \frac{\cos^2 \omega^2}{R_1} - ds_1^2 - \frac{\sin^2 \omega_0}{R_2} - ds_2^2 \right) (R_1 - R_2). \quad (7.25)$$

§26. As $ds_1, ds'_1$, are diagonals on an equilateral quadrangle, which are formed from arcs $ds_0$ of two principal tangent curves of $C$, then

$$ds_1^2 = 2ds_0^2(1 + \cos \Omega),$$

if $\Omega$ is the angle at $p$ between those two principal tangent curves. From the equation of "Dupins" indicatrix:

$$\frac{x^2}{R_1} + \frac{y^2}{R_2} = 1$$

whose asymptotes are directions of those two principal tangent curves at $p$, it follows that:

$$\cos \Omega = \frac{R_1 + R_2}{R_1 - R_2}. \quad (7.26)$$
And if $\Omega'$ is the angle, which the corresponding principal tangent curves of $C'$ (§19) form with each other on $\pi$ then formula (18) become

$$ds^2 = 2ds_0^2(1 + \cos \Omega').$$

Thus now:

$$ds^2 = ds_0^2 \frac{1 + \cos \Omega'}{1 + \cos \Omega},$$

and consequently, according to (7.23) and (7.26):

$$2 \cos^2 \omega_0 = 1 + \cos \Omega',$$

that is

$$\cos \Omega' = \cos 2\omega_0 \quad (7.27)$$

§27. From this appears simple expressions for $R'_1, R'_2$, principal radii of curvature on $\pi$ of the surface $C''$. In accordance with (7.26) we get:

$$\cos \Omega' = \frac{R'_1 + R'_2}{R'_1 - R'_2}$$

and therefore from (7.27):

$$\cos 2\omega_0 = \frac{R'_1 + R'_2}{R'_1 - R'_2},$$

we get an equation, which together with equation:

$$m^2 = -R'_1 R'_2,$$

leaves:

$$R'_1 = \pm m \cot \omega_0,$$
$$R'_2 = \pm m \tan \omega_0. \quad (7.28)$$

§28. From the surface $C$ by a parallel transformation:

$$R = R_1 + \epsilon, \ R' = R_2 + \epsilon, \ \epsilon = m\sqrt{-1} \quad (7.29)$$

a surface is formed whose principal radii of curvature are $R, R'$. I denote this surface by $\Gamma$. Then

$$\frac{1}{R} + \frac{1}{R'} = \frac{1}{m\sqrt{-1}}. \quad (7.30)$$
Surfaces $C'$ are derived from $C$ by the transformation (3.1). And surfaces (7.29) are formed from surfaces $C'$ by the same transformation (3.1), which are parallel to the special $C'$, whose surfaces I denote $\Gamma'$. Even these surfaces satisfy equation (7.30).

The elements $ds_1, ds_2$ which start from the point $p$ and which belong to two curvature curves for $C$, are transferred to elements $d\bar{s}_1, d\bar{s}_2$ of two curvature curves for $\Gamma$, and we obtain:

$$d\bar{s}_1 = \frac{R_1 + m\sqrt{-1}}{R_1} ds_1, \quad d\bar{s}_2 = \frac{R_2 + m\sqrt{-1}}{R_2} ds_2.$$ 

Thus for an arbitrary line element $d\bar{s}$ on $\Gamma$:

$$d\bar{s}^2 = d\bar{s}_1^2 + d\bar{s}_2^2 = \left( \frac{R_1 + m\sqrt{-1}}{R_1} \right)^2 ds_1^2 + \left( \frac{R_2 + m\sqrt{-1}}{R_2} \right)^2 ds_2^2,$$

or $m\sqrt{-1} = \sqrt{R_1R_2}$:

$$d\bar{s}^2 = (\sqrt{R_1} + \sqrt{R_2})^2 \left( \frac{ds_1^2}{R_1} + \frac{ds_2^2}{R_2} \right). \quad (7.31)$$

By transformations (3.1) and (7.29) we obtain from $d\bar{s}$ the element $d\bar{s}'$ on $\Gamma'$:

$$d\bar{s}'^2 = \left( \sqrt{R_1'} + \sqrt{R_2'} \right)^2 \left( \frac{ds_1'^2}{R_1'} + \frac{ds_2'^2}{R_2'} \right),$$

there $ds'_1, ds'_2$ are decided by $ds_1, ds_2$ by equations (7.23) and (7.24), consequently:

$$d\bar{s}'^2 = (R_1 - R_2) \left( \sqrt{R_1'} + \sqrt{R_2'} \right)^2 \left( \frac{ds_1^2 \cos^2 \omega_0}{R_1} - \frac{ds_2^2 \sin^2 \omega_0}{R_2} \right),$$

or because of (7.28), we can write as

$$\frac{\cos^2 \omega_0}{R_1} = \frac{1}{R_1' - R_2'}, \quad \frac{\sin^2 \omega_0}{R_2'} = \frac{1}{R_1' - R_2'},$$

$$d\bar{s}'^2 = \frac{R_1 - R_2}{R_1' - R_2'} \left( \sqrt{R_1'} + \sqrt{R_2'} \right)^2 \left( \frac{ds_1^2}{R_1} + \frac{ds_2^2}{R_2} \right). \quad (7.32)$$

From equations (7.31) and (7.32) it follows:

$$\left( \frac{d\bar{s}'}{d\bar{s}} \right)^2 = \frac{\sqrt{R_1'} + \sqrt{R_2'}}{\sqrt{R_1'} - \sqrt{R_2'}} \frac{\sqrt{R_1} + \sqrt{R_2}}{\sqrt{R_1} - \sqrt{R_2}}$$

thus:

*Each one of the surfaces $\Gamma'$ become by the transformations (7.29) and (3.1) conformably mapped on the surface $\Gamma$.***
A theorem of O. Bonnet about surfaces of constant mean curvature.

§29. Ossian Bonnet has proved that, every surface of constant mean curvature:

\[
\frac{1}{R} + \frac{1}{R'} = \frac{1}{C},
\]

(8.33)

can be transferred by the deformation into an infinite nearby surface with the same mean curvature 1/c, without having to be dilated or contracted. Therefore every surface (8.33) will be a component of a simple infinite family of similar surfaces, all applicable on each other. If we apply this theorem to the surfaces (7.30), then we obtain, by using parallel transformation, one transformation, transferring each surface of constant curvature into a number of families ∞(1) of surfaces with the same curvature. There is a big difference between transformations (3.1) and the last transformation, the surface which is transformed on the last transformation will belong to its corresponding family of surfaces, so is not the case with transformation (3.1).

I will prove Bonnet’s theorem and I will derive equations by which the Bonnet transformation is given. Those equations go from a given surface (8.33) to an infinite nearby surface of the same nature.

§30. Shall the infinitesimal transformation:

\[
\delta x = \epsilon A(x, y), \quad \delta y = \epsilon B(x, y), \quad \delta z = \epsilon C(x, y),
\]

(8.34)

when it applies to the surface:

\[
Z = f(x, Y)
\]

(8.35)

the line elements of this surface leave unchanged, so that it does not change their arc lengths, then it must satisfy:

\[
ds\delta ds = dx d\delta x + dy d\delta y + dz d\delta z = \epsilon(dAdx + dBdy + dCdz) = 0,
\]

simply

\[
dz = pdx + qdy.
\]

\[
(p = f'(x), \quad q = f'(y)).
\]

Thus for each value of \(dx, \ dy:\)

\[
\left(\frac{dA}{dx} + p \frac{dC}{dx}\right) dx^2 + \left(\frac{dB}{dy} + p \frac{dC}{dy} + q \frac{dC}{dx}\right) dy dx + \left(\frac{dB}{dy} + q \frac{dC}{dy}\right) dy^2 = 0,
\]
of which
\[ \frac{dA}{dx} + p \frac{dC}{dx} = 0, \]
\[ \frac{dB}{dy} p^2 - \left( \frac{dB}{dx} + \frac{dA}{dy} \right) pq + \frac{dA}{dx} q^2 = 0, \]
\[ \frac{dB}{dy} + \frac{dC}{dy} = 0. \]  \tag{8.36}

This equation system is the sufficient condition for the transformation (8.34) to transfer the surface (8.35) into the same applicable surface. From equation (8.36), by eliminating \( A \) and \( B \) we obtain for \( C \) this equation:
\[ t \frac{d^2C}{dx^2} - 2s \frac{d^2C}{dydx} + r \frac{d^2C}{dy^2} = 0. \]  \tag{8.37}

A and B serve as solutions of \( C = C(x, y) \), which are developed as follows. From the first equation of (8.36) we get \((dA)/(dx)\):
\[ \frac{dA}{dx} = -p \frac{dC}{dx}, \]  \tag{8.38}
and \((dA)/(dy)\) is determined by equations:
\[ \frac{d^2A}{dydx} = -p \frac{d^2C}{dx dy} - s \frac{dC}{dx}, \quad \frac{d^2A}{dy^2} = -p \frac{d^2C}{dx^2} - t \frac{dC}{dx}, \]  \tag{8.38'}
which is obtained from (36) by eliminating \( B \). Then we integrate:
\[ \int \left( \frac{dA}{dx} dx + \frac{dA}{dy} dy \right). \]

For \( B \) we have equations:
\[ \frac{dB}{dy} = -q \frac{dC}{dy}, \]  \tag{8.39}
\[ \frac{d^2B}{dx^2} = -q \frac{d^2C}{dx^2} - r \frac{dC}{dy}, \quad \frac{d^2B}{dydx} = -q \frac{d^2C}{dx dy} - s \frac{dC}{dy}, \]  \tag{8.39'}

§31. By the infinitesimal transformation (8.34) \( p, s \) are transformed into \( p + \delta p, q + \delta q \) there
\[ \delta p = \epsilon \left( \frac{dC}{dx} - p \frac{dA}{dx} - q \frac{dB}{dx} \right); \]
\[ \delta q = \epsilon \left( \frac{dC}{dy} - p \frac{dA}{dy} - q \frac{dB}{dy} \right); \]  \tag{8.40}
and \( r, s, \) and \( t \) transforms into \( r + \delta r, s + \delta s, t + \delta t \):

\[
\delta r = \epsilon \left[ (1 + p^2 + q^2) \frac{d^2 C}{dx^2} + r \left( \frac{dC}{dx} + q \frac{dC}{dy} \right) - 2 \left( \frac{dA}{dx} + s \frac{dB}{dx} \right) \right],
\]

\[
\delta s = \epsilon \left[ (1 + p^2 + q^2) \frac{d^2 C}{dxdy} + s \left( \frac{dC}{dx} + q \frac{dC}{dy} \right) \right. \]
\[
- \left. \left( r \frac{dA}{dy} + s \left( \frac{dA}{dx} + \frac{dB}{dy} \right) + t \frac{dB}{dx} \right) \right],
\]

\[
\delta t = \epsilon \left[ (1 + p^2 + q^2) \frac{d^2 C}{dy^2} + t \left( \frac{dC}{dx} + q \frac{dC}{dy} \right) - 2 \left( s \frac{dA}{dy} + t \frac{dB}{dy} \right) \right].
\]

\[\text{§32.}\] The principal curvature radii \( R, R' \) of (8.35) are roots of the equation:

\[R^2(rt - s^2) - R\sqrt{1 + p^2 + q^2}[(1 + q^2)r - 2pq + (1 + p^2)t] + (1 + p^2 + q^2)^2 = 0.\]

Because of (8.36), (8.40), (8.41) we obtain:

\[\delta(1 + p^2 + q^2) = 2\epsilon(1 + p^2 + q^2) \left( \frac{dC}{dx} + q \frac{dC}{dy} \right),\]

\[\delta(rt - s^2) = \epsilon(1 + p^2 + q^2) \left( t \frac{d^2 C}{dx^2} - 2s \frac{d^2 C}{dxdy} + r \frac{d^2 C}{dy^2} \right) + 4\epsilon(rt - s^2) \left( \frac{dC}{dx} + q \frac{dC}{dy} \right),\]

of which, observing equation (8.37):

\[\delta \left( \frac{rt - s^2}{(1 + p^2 + q^2)^2} \right) = 0.\]

I.e. the curvature \( 1/(RR') \) will not change by deformation (Gauss theory). Furthermore:

\[\delta[(1 + p^2)r - 2pq + (1 + p^2)t] = \epsilon(1 + p^2 + q^2) \left[ (1 + q^2) \frac{d^2 C}{dx^2} - 2pq \frac{d^2 C}{dxdy} + (1 + p^2) \frac{d^2 C}{dy^2} \right] \]

\[+ 3\epsilon \left( \frac{dC}{dx} + q \frac{dC}{dy} \right) [(1 + q^2)r - 2pq + (1 + p^2)t].\]

Hence:

\[\delta \left( \frac{1}{R} + \frac{1}{R'} \right) = \frac{\epsilon}{\sqrt{1 + p^2 + q^2}} \left[ (1 + q^2) \frac{d^2 C}{dx^2} - 2pq \frac{d^2 C}{dxdy} + (1 + p^2) \frac{d^2 C}{dy^2} \right].\]

\[\text{§33.}\] If thus \( R, R' \) will be unchanged by the deformation of the surface (8.35), then both equations (8.37) and

\[(1 + q^2) \frac{d^2 C}{dx^2} - 2pq \frac{d^2 C}{dxdy} + (1 + p^2) \frac{d^2 C}{dy^2} = 0\] (8.42)
must possess one common solution $C = C(x, y)$. By eliminating \((d^2 C)/(dy^2)\) and \((d^2 C)/(dx^2)\) we get:

\[
\frac{d^2 C}{dx^2}[(1 + p^2)t - (1 + q^2)r] = 2\frac{d^2 C}{dx dy}[(1 + p^2)s - pqr],
\]

\[
\frac{d^2 C}{dy^2}[(1 + p^2)t - (1 + q^2)r] = 2\frac{d^2 C}{dx dy}[pqt - (1 + p^2)s].
\]

If we write:

\[
X = \frac{p}{\sqrt{1 + p^2 + q^2}}, \quad Y = \frac{q}{\sqrt{1 + p^2 + q^2}}
\]

then we have:

\[
\frac{(1 + p^2)t - (1 + q^2)r}{(1 + p^2 + q^2)^{\frac{3}{2}}} = \frac{dY}{dy} - \frac{dX}{dx},
\]

\[
\frac{(1 + p^2)s - pqr}{(1 + p^2 + q^2)^{\frac{3}{2}}} = \frac{dY}{dx},
\]

\[
\frac{(1 + p^2)s - pqt}{(1 + p^2 + q^2)^{\frac{3}{2}}} = \frac{dX}{dy},
\]

and consequently equations (8.37), (8.42) are replaced by the equation system:

\[
\frac{d^2 C}{dx^2} = \lambda \frac{dY}{dx},
\]

\[
2\frac{d^2 C}{dx dy} = \lambda \left( \frac{dY}{dx} - \frac{dX}{dx} \right),
\]

\[
\frac{d^2 C}{dy^2} = -\lambda \frac{dX}{dy}.
\]

(8.43)

When $\lambda$ is constant, independently from $x$ and $y$, then it is necessary that

\[
\frac{dX}{dx} + \frac{dY}{dy},
\]

\[
\frac{1}{R} + \frac{1}{R'} = \frac{1}{c},
\]

must be constant, independent from $x$ and $y$ so that surface (8.35) will satisfy one equation (8.33).

Thus the pair of equations (8.37), (8.42), equivalent to (8.43), have a common solution $C = C(x, y)$, when the following relation

\[
\frac{1}{R} + \frac{1}{R'} = \frac{1}{c},
\]
applies for the principal curvature radius of the surface (8.35) and the same surface will be transferred by the transformation

\[ \delta x = \epsilon A(x, y), \quad \delta y = \epsilon B(x, y), \quad \delta z = \epsilon C(x, y), \]

into an infinite nearby surface with the same constant mean curvature, in case A, B, C are determined as follows. From (8.43), there is a constant \( \lambda \) such that

\[ \frac{dC}{dx} = \lambda \left( Y - \frac{1}{2} \frac{xdx - ydy}{\sqrt{1 + p^2 + q^2}} - \frac{1}{2c} (yd - xdx) \right) + \lambda [k_1x + k_2y]. \]

For the determination of A and B we refer to the procedure indicated in the last part of §30. From (8.38) we get:

\[ \frac{dA}{dx} = -\lambda [YP - \frac{1}{2c}p + k_1p], \]

from (38'):

\[ \frac{dA}{dy} = \lambda \left[ -\frac{1}{\sqrt{1 + p^2 + q^2}} + pX + \frac{1}{2c}q - \frac{1}{2c}k_1p + k \right], \]

from (39'):

\[ \frac{dB}{dx} = \lambda \left[ -\frac{1}{\sqrt{1 + p^2 + q^2}} - qY - \frac{1}{2c}p + \frac{1}{2c}k_2p + k' \right], \]

from (8.39):

\[ \frac{dB}{dy} = -\lambda \left[ \frac{1}{2c}q - Xq + k_2q \right]. \]

Here nevertheless according to the second equation of (8.36):

\[ k = k'. \]

To put constants \( k_1, k_2 \) equal to zero, is the same as ignoring one arbitrary infinitesimal rotation of the surface (8.35). Ignoring also the last integration in order to get three consonants A, B, B, is the same as ignoring an arbitrary infinitesimal translation of this surface. \( \lambda \) can be added to \( \epsilon \), so that in previous formulae we can put \( \lambda = 1 \).
9 Notation I. Proof of formula (B) in §2

Values of \( p' \) and \( q' \) which are derived by the equations:

\[
F_3'(z', x'y', p', q') = 0,
\]

\[
F_4'(z', x'y', p', q') = 0,
\]

makes \( dz' - p'dx' - q'dy' = 0 \) to an exact differential of an equation: \( \varphi(x', y', z') = 0 \) to an arbitrary constant. The condition for this is given by the equation:

\[
[F_3'F_4'] = \left( \frac{\partial F_3'}{\partial x'} + p' \frac{\partial F_3'}{\partial z'} \right) \frac{\partial F_4'}{\partial q'} - \left( \frac{\partial F_4'}{\partial x'} + p' \frac{\partial F_4'}{\partial z'} \right) \frac{\partial F_3'}{\partial q'} = 0.
\]

Thus if the transformation (A) shall transfer the surface \( z = f(x, y) \) into a family of surfaces: \( \varphi(x', y', z') = 0 \) to an arbitrary constant, then the equation \( [F_3'F_4'] = 0 \) must be satisfied, when equations which are obtained by elimination of \( z, x, y, p, q \) between (A) and \( z = f(x, y), p = f'(x), q = f'(y) \), are denoted by

\[
F_3'(z', x', y', p', q') = 0, \quad F_4'(z', x', y', p', q') = 0.
\]

If we use equations \( z = f(x, y), p = f'(x), q = f'(y) \), together with equations \( F_1 = 0, \quad F_2 = 0 \) to determine \( z, x, y, p, q \) on the function of \( z', x', y', p', q' \), after substituting the values for \( z, x, y, p, q \) on \( F_3, F_4 \) we obtain \( F_3', F_4' \). When we write:

\[
[F_iF_k]_{z',x'} = \left( \frac{\partial F_i}{\partial x'} + p' \frac{\partial F_i}{\partial z'} \right) \frac{\partial F_k}{\partial q'} + \left( \frac{\partial F_i}{\partial y'} + q' \frac{\partial F_i}{\partial z'} \right) \frac{\partial F_k}{\partial q'}
\]

for \( i, k = 1, 2, 3, 4; \quad p = f'(x), q = f'(y), \quad r = \frac{\partial^2 f}{\partial x^2}, \quad s = \frac{\partial^2 f}{\partial x\partial y}, \quad t = \frac{\partial^2 f}{\partial y^2} \)

then we get:

\[
[F_3'F_4'] = [F_3F_4]_{z',x'} + \frac{dF_3}{dx}[xF_4] + \frac{dF_3}{dy}[yF_4] + \frac{dF_4}{dx}[F_3x] + \frac{dF_4}{dx}[F_3y] + \frac{dF_4}{dy}[F_3y] - \frac{dF_4}{dx}[F_3y].
\]
We replace \( z, p, q \) in \( F_1, F_2, F_3, F_4 \) by \( f, f'(x), f(y) \) and thereafter we replace \( x, y \) in \( F_1 = 0, F_2 = 0 \) by \( z', x', y', p', q' \). If we denote by \( F_m' \) the expression on \( z', x', y', p', q' \), then we have \( F'_1, F'_2 \) identically equal to 0. Therefore:

\[
\begin{align*}
[F'_1 F_3] &= 0 = [F_1 F_3]_{x'p'} + \frac{dF_1}{dx}[xF_3] + \frac{dF_3}{dy}[yF_3], \\
[F'_2 F_3] &= 0 = [F_2 F_3]_{x'p'} + \frac{dF_2}{dx}[xF_3] + \frac{dF_3}{dy}[yF_3], \\
[F'_1 F_4] &= 0 = [F_1 F_4]_{x'p'} + \frac{dF_1}{dx}[xF_4] + \frac{dF_4}{dy}[yF_4], \\
[F'_2 F_4] &= 0 = [F_2 F_4]_{x'p'} + \frac{dF_2}{dx}[xF_4] + \frac{dF_4}{dy}[yF_4], \\
[F'_1 F_2] &= 0 = [F_1 F_2]_{x'p'} + \frac{dF_1}{dx}[xF_2] + \frac{dF_2}{dy}[yF_2], \\
[F'_2 x] &= 0 = [F_2 x]_{x'p'} + \frac{dF_2}{dy}[yx], \\
[F'_2 y] &= 0 = [F_2 y]_{x'p'} + \frac{dF_2}{dx}[xy].
\end{align*}
\]

By eliminating \([xF_3], \ldots, [xy] \) we get, when we write

\[
(i, k) \text{ for } \frac{dF_i}{dx} \frac{dF_k}{dy} - \frac{dF_i}{dy} \frac{dF_k}{dx},
\]

\[
(4.12) [F'_3 F'_4] = (8.34)[F_1 F_2]_{x'p'} + (8.42)[F_1 F_3]_{x'p'} + (7.23)[F_1 F_4]_{x'p'}
\]

\[
+ (4.12)[F_3 F_4]_{x'p'} + (4.13)[F_4 F_2]_{x'p'} + (5.14)[F_2 F_3]_{x'p'}.\]

The condition \([F'_3 F'_4] = 0 \) is expressed by the equation (B).

\section{Notation II. Three examples of transformations (A)}

These transformations are determined by four equations of the form (A) but there \( z \) and \( z' \) is missing. These equations have the form:

\[
\begin{align*}
x' &= f_1(x, y, p, q), \\
y' &= f_2(x, y, p, q), \\
p' &= f_3(x, y, p, q), \\
q' &= f_4(x, y, p, q),
\end{align*}
\]

there \( f_1, f_2, f_3 \) and \( f_4 \) are algebraic functions of \( x, y, p, q \). The conditional equation (B) now takes this form:

\[
(4.13) + (7.24) = 0.
\]
and becomes consequently an equation of second order in the domain \( r \), of this type:

\[ Ar + Bs + Ct + D(rt - s^2) + E = 0, \]

there \( A, B, C, D \) and \( E \) are functions of \( x, y, p, q \). The integral surface of this equation are those surfaces on \( r \), which are transformed into surfaces on \( r' \).

Those surfaces on \( r' \) are integrals to an partial differential equation of the second order of the same type as (a). An integral surface on \( r \), together with all integrals which are obtained by the translation outwards the Z-axis, correspond to an integral surface of an equation on \( r' \), together with all integrals, which are obtained from the translation of the surface outwards the Z-axis.

Through quadratures only we obtain those surfaces on one domain that correspond to a given surface on the other domain.

1. Transformation:

\[ x' = ap, \ y' = aq, \ p' = \frac{y}{a}, \ q' = -\frac{x}{a} \]

transfers integral surfaces of the equation:

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

into integral surfaces for the same equation.

2. Transformation:

\[ x' = \sqrt{-1} y, \ y' = -\sqrt{-1} x, \ p' = \sqrt{-1} \frac{p}{\sqrt{1 + p^2 + q^2}}, \ q' = \sqrt{-1} \frac{p}{\sqrt{1 + p^2 + q^2}} \]

transfers every minimal surface yet again into a minimal surface.

3. Transformation:

\[ x' = a \frac{p}{\sqrt{1 + p^2 + q^2}}, \ y' = x, \ p' = \frac{q}{\sqrt{1 + p^2 + q^2}}, \ q' = \frac{y}{a} \]

transfers surfaces of constant curvature \( -1/a^2 \) on domain \( r \) into integrals of differential equation:

\[ (a^2 - x'^2)(r't' - s'^2) + p'x'(r' + t') + 1 - p'^2 = 0 \]

Since this equation will not change by the dual transformation: \( x' = ap'', \ y' = aq'', \ ap' = x'', \ aq' = y'' \), then every surface with curvature curve \( -1/a^2 \) of the corresponding transformation on \( r \) is transferred into a similar surface. The corresponding transformation on \( r \) is represented by equations: \( x = y'' \), \( y = x'' \), \( p = q'' \), \( q = p'' \), and is realized by a rotation \( 90^\circ \) around Z-axis, followed by a reflection toward \( YZ \)-plane. This is a transformation which leaves the curvature unchanged.
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Bibliography


Survey of mathematical models in biology from point of view of Lie group analysis

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Abstract. This is an attempt to apply methods from Lie group analysis to models from mathematical biology. In the first part there is a collection of mathematical models from biology with brief descriptions. Some of these models are then analyzed using Lie group analysis. The results from calculations of Lie point symmetries and equivalence symmetries are contained in one part. The other part contains the results from calculations of invariant solutions.
I Models from biology

1 Models formulated in terms of ordinary differential equations of the first order

1.1 Population models

1.1.1 Introduction

The basic assumption for the models in this section is that the populations are considered to be homogeneous, this means that each individual in the population is considered to be identical to the others. One way to overcome this is to divide the population into different classes. Also her is only continuous models. The population models are considering both as isolated populations and as interaction between populations.

1.1.2 Simple continuous population models

A simple continuous population model for a single species, with \( N(t) \) as the population of the species at time \( t \). Here the rate of change of \( N \) is formed by birth rate minus death rates. The governing equation for \( N(t) \) is (see e.g. [7], chapter 1)

\[
\frac{dN}{dt} = bN - dN
\] (1.1)

where \( b, d \) are positive constants. The term \( bN \) represents the birth rate where \( b \) is birth rate per individual. The term \( -dN \) represents the death rate and \( d \) the per capita death rate. The solution of (1.1) is given by \( N = Ce^{(b-d)t} \), so here \( N \) either grow exponentially or decay exponentially depending on if \( b > d \) or \( d > b \). More generally with arbitrary population dependent birth and death rates, the equation for \( N \) is given by

\[
\frac{dN}{dt} = (b(N) - d(N))N.
\] (1.2)

A modification of the exponential growth to restricted growth was suggested by Verhulst (1838,1845), this he called logistic growth. Instead of a population independent per capita growth rate as in (1.1) here the per capita growth rate depend linearly
on the population. The equation is given by (see e.g. [7], chapter 1)

\[ \frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right) \]  

(1.3)

where \( r, K \) are positive constants, \( r \) is called the linear reproduction rate. Now there is a self-limiting process, because if \( N > K \) then \( dN/dt < 0 \) and if \( N < K \) then \( dN/dt > 0 \). The constant \( K \) is called the carrying capacity of the environment. It can be shown that \( N = K \) is a stable steady state of (1.3), so \( K \) determines the size of the stable steady state population, see [7].

The logistic growth model could be modified by adding a term representing harvesting, we then have the equation

\[ \frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right) - EN \]  

(1.4)

where \( E \) is a positive constant.

1.1.3 Population model for spruce budworms

A population model for spruce budworms. The spruce budworm is an insect which lives in balsam firs and cause damage by devouring the foliage of the balsam firs. With \( N \) as the population density, the equation for the model could be written as (see e.g. [7], chapter 1)

\[ \frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right) - p(N) \]  

(1.5)

where \( r, K \) are constants and \( p(N) \) is a function which represents predation. Typically \( p(N) \) is a monotonic increasing function which saturates for large \( N \). A specific form for \( p(N) \) suggested by Ludwig et al. (1978) is given by \( p(N) = BN^2/(A^2 + N^2) \).

1.1.4 Interacting species

Population models for interacting species. For two-species one can distinguish between the following three types of interactions: predator-prey interaction where the growth rate of one population (prey) is decreased while the growth rate of the other population (predator) increase, competition interaction where both growth rates are decreased and mutualism (symbiosis) interaction where both growth rates are increased.

A model for two interacting species proposed by Volterra (1926). It was also proposed by Lotka (1920) for a chemical reaction, and its called Lotka-Volterra model. Let \( N(t) \) denote the prey population and \( P(t) \) the predator population at time \( t \). The assumptions for the model are: (i) the birth rate for the prey is proportional to its population, (ii) each predator catch prey at a rate proportional to the prey population, (iii)
the per capita birth rate of the predator is proportional to the prey population, (iv) that
the predator die at rate proportional to its population. The predator-prey interaction is
then modelled by the equations (see e.g. [7], chapter 3)

\[
\frac{dN}{dt} = N(a - bP), \\
\frac{dP}{dt} = P(cbN - d),
\]

(1.6)

where \(a, b, c, d\) are positive constants. The term \(bN\) can be interpreted as the number
of prey eaten in unit time per predator.

A more general predator-prey model where \(N(t)\) and \(P(t)\) denote the population
of the prey respectively predator at time \(t\). By taking the prey to satisfy logistic growth
in the absence of any predator, with a density dependent predation function \(R(N)\) and
a general per capita growth rate \(G(N, P)\) for the predator. The equations now becomes
(see e.g. [7], chapter 3)

\[
\frac{dN}{dt} = N \left[ r \left( 1 - \frac{N}{K} \right) - PR(N) \right], \\
\frac{dP}{dt} = PG(N, P).
\]

(1.7)

where \(r, K\) are positive constants. \(NR(N)\) can be interpreted as the number of prey
lost in unit time to each predator. If increasing prey leads to increased lost of prey up to
some level, then \(NR(N)\) should increase with \(N\) and saturate for large \(N\). Suggested
forms for predation function \(R(N)\) are given by (see [7])

\[ R(N) = \frac{A}{N + B}, \quad R(N) = \frac{AN}{N^2 + B^2}, \quad R(N) = \frac{A(1 - e^{-aN})}{N}, \]

and suggested forms for \(G(N, P)\) are given by (see [7])

\[ G(N, P) = k \left( 1 - \frac{hP}{N} \right), \quad G(N, P) = -d + qR(N), \]

where \(A, B, a, k, h, d, q\) are positive constants.

Another possible predator-prey model is given by the equations

\[
\frac{dN}{dt} = N[r f(N) - d - PR(N)], \\
\frac{dP}{dt} = PG(N, P),
\]

(1.8)

where \(r, d\) are positive constants. Here \(r f(N)\) is the per capita birth rate for prey, \(d\) the
per capita natural death rate for prey and each predator kill prey at the rate \(NR(N)\).
\(R(N)\) and \(G(N, P)\) can have the same form as in (1.7).
A 2-species competition model where \( N_1, N_2 \) denotes population density. Each species is taken to have logistic growth in the absence of the other. The equations are written in dimensionless form as (see e.g. [7], chapter 3)

\[
\frac{dN_1}{dt} = r_1 N_1 \left( 1 - \frac{N_1}{K_1} - b_{12} \frac{N_2}{K_1} \right), \\
\frac{dN_2}{dt} = r_2 N_2 \left( 1 - \frac{N_2}{K_2} - b_{21} \frac{N_1}{K_2} \right),
\]

where \( r_1, r_2, K_1, K_2, b_{12}, b_{21} \) are positive constants. \( r_1, r_2 \) and \( K_1, K_2 \) are the linear birth rate and carrying capacity for the first respectively the second species. If \( b_{12}, b_{21} \) are negative constants we have a mutualism model.

A three species model from Lorenz (1963), written as (see e.g. [7], chapter 3)

\[
\frac{du}{dt} = a(u - v), \quad \frac{dv}{dt} = uw + bu - v, \quad \frac{dw}{dt} = uw - cw,
\]

where \( a, b, c \) are positive constants.

### 1.2 Reaction kinetics

#### 1.2.1 Introduction

In this section we deal with models of biochemical reactions. Reaction kinetics is concerned with the rates of change in the concentration of reactants in chemical reactions. A reaction mechanism can schematically be described as

\[
A + B \xrightleftharpoons[k_{-1}]{k_1} C \rightarrow D,
\]

where \( A, B, C, D \) represents chemical compounds and \( k_1, k_{-1}, k_2 \) are constants associated with the reaction rate. The symbol \( \xrightarrow{\Rightarrow} \) indicates that reactions can go both ways and the symbol \( \rightarrow \) that the reaction only can go one way. A more general reaction mechanism is given by

\[
\alpha A + \beta B \xrightleftharpoons[k_{-1}]{k_1} \lambda C
\]

where \( \alpha, \beta, \lambda \) are stoichiometric factors, this means for example that \( \alpha \) number of \( A \) and \( \beta \) number of \( B \) are used in the reaction to produce \( \lambda \) number of \( C \). To denote concentrations of the compound one usually enclose them with brackets, so for example \([A]\) denotes the concentration of \( A \). The law of mass action is used to describe the rate of reactions mathematically. The law of mass action says that the rate of a reaction is
proportional to the product of the concentrations of the reactants. Applying this to the mechanism (1.12) gives the equations

\[
\begin{align*}
\frac{d[A]}{dt} &= -\alpha k_1 [A]^\alpha [B]^\beta + \alpha k_{-1} [C]^\lambda, \\
\frac{d[B]}{dt} &= -\beta k_1 [A]^\alpha [B]^\beta + \beta k_{-1} [C]^\lambda, \\
\frac{d[C]}{dt} &= \lambda k_1 [A]^\alpha [B]^\beta - \lambda k_{-1} [C]^\lambda.
\end{align*}
\]

(1.13)

### 1.2.2 Basic enzymatic reaction

Looking at a basic enzymatic reaction. This reaction includes an enzyme \(E\), substrate \(S\), the complex \(SE\) and the product \(P\). This reaction was first proposed by Michaelis and Menten (1913). The reaction mechanism is represented schematically by (see e.g. [7], chapter 6)

\[
S + E \xrightarrow{k_1} SE \xrightarrow{k_2} P + E
\]

(1.14)

where \(k_1, k_{-1}, k_2\) are reaction constants. It can be described as one molecule of the substrate \(S\) combines with one molecule of the enzyme \(E\) to form one of \(SE\), from here there is the possible to either produce one molecule of the product \(P\) and \(E\) or go back to \(S\) and \(E\). Let’s use lowercase letters \(s, e, p\) for denoting the concentrations of \(S, E, P\) and use \(c\) for the concentration of the complex \(SE\). Applying the Law of Mass Action we end up with the system of equations

\[
\begin{align*}
\frac{ds}{dt} &= -k_1 es + k_{-1} c, \\
\frac{de}{dt} &= -k_1 es + (k_{-1} + k_2)c, \\
\frac{dc}{dt} &= k_1 es - (k_{-1} + k_2)c, \\
\frac{dp}{dt} &= k_2 c
\end{align*}
\]

(1.15)

where all \(k_i\) are constants. Appropriate initial conditions could be

\[s(0) = s_0, \ e(0) = e_0, \ c(0) = 0, \ p(0) = 0.\]

Now trying to simplify the system (1.15). By adding the reaction equations for \(e\) and \(c\) of (1.15), integration and use of initial conditions yields that \(e(t) + c(t) = e_0\). Now since \(p\) and \(e\) can be solved once \(c\) is found, the system (1.15) can be reduced to

\[
\begin{align*}
\frac{ds}{dt} &= -k_1 e_0 s + (k_1 s + k_{-1})c, \\
\frac{dc}{dt} &= k_1 e_0 s - (k_1 s + k_{-1} + k_2)c.
\end{align*}
\]

(1.16)
Assuming that after the initial stage of the reaction we have $dc/dt \approx 0$, see [7]. Then setting $dc/dt = 0$ in (1.16) and solve the second equation in (1.16) for $c$, which gives

$$c(t) = \frac{e_0s(t)}{s(t) + K_m}, \quad \text{where} \quad K_m = \frac{k_{-1} + k_2}{k_1}.$$ 

Substituting this into the first equation in (1.16) and the equation is now

$$\frac{ds}{dt} = -\frac{k_2e_0s}{s + K_m},$$

where the constant $K_m$ is called the Michaelis constant.

1.2.3 Suicide substrate

A reaction called a suicide substrate system or mechanism-based inhibitor. It involves an enzyme $E$, substrate $S$, product $P$, enzyme-substrate intermediates $X, Y$ and inactivated enzyme $E_i$. The mechanism is given by (see e.g. [7], chapter 6)

$$E + S \xrightarrow{k_1} X \xrightarrow{k_2} Y \xrightarrow{k_3} E + P \quad \xrightarrow{k_4} E_i$$

where the $k$’s are positive rate constants. $S$ is called suicide substrate since after binding with the enzyme there is the possibility for the enzyme to be irreversibly inactivated (by the transition to $E_i$). So the substrate after binding with the enzyme could have the role of an inhibitor. With sufficient amount of suicide substrate it could be possible to target a specific type of enzyme for inactivation. With lowercase letters denoting concentrations and using the law of mass action yields the following system of equations

\begin{align*}
\frac{ds}{dt} &= -k_1es + k_{-1}x, \\
\frac{dc}{dt} &= -k_1es + k_{-1}x + k_3y, \\
\frac{dx}{dt} &= k_1es - k_{-1}x - k_2x, \\
\frac{dy}{dt} &= k_2x - k_3y - k_4y, \\
\frac{de_i}{dt} &= k_4y, \\
\frac{dp}{dt} &= k_3y.
\end{align*}

(1.17)
Typical initial conditions are given by

\[
e(0) = e_0, \quad s(0) = s_0, \quad x(0) = y(0) = e_i(0) = p(0) = 0
\]

Now trying to simplify the system (1.17). Since \( p \) don’t exist in any of the equations in (1.17). Then from the last equation in (1.17), \( p \) can be evaluated by integration once \( y \) is known. By adding the 2nd-5th equations in (1.17) and then integrating and using the initial conditions yields \( e + x + y + e_i = e_0 \). Now we can reduce the system (1.17) to

\[
\begin{align*}
\frac{ds}{dt} &= -k_1(e_0 - x - y - e_i)s + k_{-1}x, \\
\frac{dx}{dt} &= k_1(e_0 - x - y - e_i)s - (k_{-1} + k_2)x, \\
\frac{dy}{dt} &= k_2x - (k_3 + k_4)y, \\
\frac{de_i}{dt} &= k_4y.
\end{align*}
\]

(1.18)

### 1.2.4 Cooperative reaction

An example of a cooperative reaction mechanism. It involves an enzyme \( E \) which has two binding sites, a substrate \( S \), two enzyme-substrate complex \( C_1, C_2 \) and a product \( P \). The reaction is called cooperative since the enzyme \( E \) after binding with the substrate \( S \) and forming the complex \( C_1 \) can bind again with the substrate \( S \) to form the complex \( C_2 \). The complex \( C_1 \) can breaks down to form enzyme \( E \) and product \( P \) and complex \( C_2 \) can break down to form \( C_1 \) and product \( P \). The mechanism is given by (see e.g. [7], chapter 6)

\[
\begin{align*}
S + E & \xrightarrow{k_1} \quad C_1 \xrightarrow{k_2} E + P, \\
S + C_1 & \xrightarrow{k_3} \quad C_2 \xrightarrow{k_4} C_1 + P
\end{align*}
\]

(1.19)

where the \( k \)'s are positive rate constants. With lowercase letters denoting concentrations, the law of mass action applied to (1.19) gives the following system of equations

\[
\begin{align*}
\frac{ds}{dt} &= -k_1se + (k_{-1} - k_3s)c_1 + k_{-3}c_2, \\
\frac{dc_1}{dt} &= k_1se - (k_{-1} + k_2 + k_3s)c_1 + (k_{-3} + k_4)c_2, \\
\frac{dc_2}{dt} &= k_3sc_1 - (k_{-3} + k_4)c_2, \\
\frac{de}{dt} &= -k_1se + (k_{-1} + k_2)c_1, \\
\frac{dp}{dt} &= k_2c_1 + k_4c_2.
\end{align*}
\]

(1.20)
Appropriate initial conditions are (see [7])

\[ s(0) = s_0, \quad e(0) = e_0, \quad c_1(0) = c_2(0) = p(0) = 0. \]

Now trying to simplify the system (1.20). Since \( p \) don’t exist in any of the equations in (1.20). Then from the last equation in (1.20), we see that \( p \) can be evaluated by integration once \( c_1, c_2 \) are known. By adding the 2nd-4th equations in (1.20), integrating and using the initial conditions yields \( e + c_1 + c_2 = 0 \). By using this, the system (1.20) is reduced to

\[
\begin{align*}
\frac{ds}{dt} &= -k_1 e_0 s + (k_{-1} + k_1 s - k_3 s) c_1 + (k_1 s + k_{-3}) c_2, \\
\frac{dc_1}{dt} &= k_1 e_0 s - (k_{-1} + k_2 + k_1 s + k_3 s) c_1 + (k_{-3} + k_4 - k_1 s) c_2, \\
\frac{dc_2}{dt} &= k_3 s c_1 - (k_{-3} + k_4) c_2.
\end{align*}
\]

### 1.2.5 Autocatalysis

Example of a reaction with autocatalysis. In autocatalysis a chemical is involved in its own production, in this example it is the chemical \( X \). The reaction mechanism is given by (see [7], chapter 6)

\[ A + X \xrightarrow{k_1} 2X \]

where the \( k \)'s are positive rate constants. With lowercase letters denoting concentrations, the law of mass action applied to (1.22) gives the following system of equations

\[
\begin{align*}
\frac{da}{dt} &= -k_1 a x + k_{-1} x^2, \\
\frac{dx}{dt} &= k_1 a x - k_{-1} x^2.
\end{align*}
\]

The reaction rate for \( x \) is inhibited by a feedback from the product (which in this case also is \( x \)). In the equation this could be seen from the term \( -k_{-1} x^2 \).

Another reaction mechanism with autocatalysis is given by (see [7], chapter 6)

\[ A + X \xrightarrow{k_1} 2X, \quad B + X \xrightarrow{k_2} C \]

where the \( k \)'s are positive rate constants. Using lower case letters for the concentration the law of mass action gives the equations

\[
\begin{align*}
\frac{da}{dt} &= -k_1 a x + k_{-1} x^2, \quad \frac{db}{dt} = -k_2 b x, \\
\frac{dx}{dt} &= k_1 a x - k_{-1} x^2 - k_2 b x, \quad \frac{dc}{dt} = k_2 b x.
\end{align*}
\]
1.2.6 Various other reactions

A model for a reaction mechanism called the Thomas (1975) mechanism, it involves the substrates \( u \) and \( v \). The equations for the reaction system is written in dimensionless form as (see e.g. [7], chapter 6)

\[
\frac{du}{dt} = a - u - \rho R(u, v), \\
\frac{dv}{dt} = \alpha(b - v) - \rho R(u, v), \\
R(u, v) = \frac{uv}{1 + u + Ku^2}.
\] (1.26)

where \( a, b, \alpha, \rho, K \) are positive constants. The model can be interpreted in the following way: \( u \) and \( v \) are supplied at the constant rates \( a \) and \( \alpha b \). \( u \) and \( v \) degraded linearly proportional to their concentrations, (the \(-u\) and \(-\alpha v\) terms), this kind of degradation is called first-order kinetics removal. \( u \) and \( v \) are used up in the reaction at the rate \( \rho R(u, v) \). Now looking at \( R(u, v) \) for a fix \( v \), for \( u < 1/\sqrt{K} \) then \( R(u, v) \) will increase with increasing \( u \). For \( u > 1/\sqrt{K} \) then \( R(u, v) \) will decreases with increasing \( u \). Because of this behaviour \( R(u, v) \) is said to exhibit substrate inhibition.

An example of an activator-inhibitor model, proposed by Gierer and Meinhardt. The equations for the reaction system is written in dimensionless form as (see e.g. [7], chapter 6)

\[
\frac{du}{dt} = a - bu + \frac{u^2}{v(1 + Ku^2)}, \\
\frac{dv}{dt} = u^2 - v.
\] (1.27)

where \( a, b, K \) are constants. Here both \( u \) and \( v \) degrade according to first-order kinetics removal, this is the terms \(-bu\) respectively \(-v\). \( u \) activates \( v \) through the term \( u^2 \). The term \( u^2/[v(1 + Ku^2)] \) can be interpreted both as autocatalytic production of \( u \) which saturates to \( 1/Ku \) for large \( u \), also it could be interpreted as inhibition from \( v \) on \( u \) since \( u^2/[v(1 + Ku^2)] \) decreases as \( v \) increases. This kind of inhibition is called feedback inhibition of \( v \) on \( u \).

A reaction model proposed by Gray and Scott. The rate equations are written in dimensionless form as (see e.g. [7], chapter 6)

\[
\frac{du}{dt} = a(1 - u) - uv^2 - bu, \\
\frac{dv}{dt} = a(c - v) + uv^2 + bu - dv.
\] (1.28)

where \( a, b, c, d \) are constants.
1.3 Biological oscillators

In this section there are models of biological oscillators. Usually the equations look like this

\[ \frac{d\mathbf{u}}{dt} = f(\mathbf{u}) \]

where \( \mathbf{u} \) is a vector and \( f(\mathbf{u}) \) describes the nonlinear reaction kinetics. It is of interest here to know if the equations admit periodic solutions. Equations are usually in [7] investigated for existence of limit cycle solutions. A limit cycle solutions is defined as a closed trajectory in the \( \mathbf{u} \) space that is not a member of a continuous family of closed trajectory.

1.3.1 Feedback mechanism

A generalization of a model from Goodwin (1965). \( M \) represent the concentrations of the mRNA, \( E \) the concentration of the enzyme and \( P \) the concentration of the product. \( P \) is the product of the reaction of the enzyme and a substrate which is assumed to be available at a constant level. The rate equations are given by (see e.g. [7], chapter 7)

\[
\begin{align*}
\frac{dM}{dt} &= V \frac{D}{D + P^m} - aM, \\
\frac{dE}{dt} &= bM - cE, \\
\frac{dP}{dt} &= dE - eP,
\end{align*}
\]

(1.29)

where \( V, m \) (the Hill coefficient) and \( a, b, c, d, e \) are positive constants. The model (1.29) can be interpreted in the following way. All three compounds degrade with first-order kinetics removal. There is a chain of activation, \( M \) activates \( E \) via the term \( bM \) and \( E \) activates \( P \) via the term \( dE \). Then there is the feedback inhibition, \( P \) inhibits the rate of \( M \) via the term \( V/(D + P^m) \). A possible modification of the system (1.29) is to have the degradation of \( P \) saturating for large \( P \). Then the equation for \( P \) is written as (see [7], chapter 7)

\[ \frac{dP}{dt} = dE - \frac{eP}{k + P} \]

where \( k \) is a constant. Limit cycle solutions can occur for equations (1.29), see [7].

1.3.2 Sequence of linked reactions

A possible model for a sequence of linked reactions, generalised to \( n \) reactions and with a general nonlinear feedback function \( f(u_n) \). The reaction system is written in
nondimensional form as (see e.g. [7], chapter 7)

\[
\begin{align*}
\frac{du_1}{dt} &= f(u_n) - k_1 u_1, \\
\frac{du_r}{dt} &= u_{r-1} - k_r u_r, \quad r = 2, 3, \ldots, n,
\end{align*}
\]

(1.30)

where \(k_1, k_r\) are positive constants and \(f(u_n) \geq 0\). If \(f'(u_n) > 0\) then the equations (1.30) represents a positive feedback loop and if \(f'(u_n) < 0\) it is a negative feedback loop also called feedback inhibition. Two possible forms for the feedback function \(f(u_n)\) are given by (see [7])

\[
\begin{align*}
f(u_n) &= a + \frac{u^n_m}{1 + u^n_m}, \\
f(u_n) &= \frac{1}{1 + u^n_m},
\end{align*}
\]

where \(a, m\) are positive constants, the first function represents a positive and the other a negative feedback control. See Yagil and Yagil (1971) for other possible forms for \(f(u_n)\).

1.3.3 EOB reaction

A reduced model of the oscillatory iodate-sulphite-ferrocyanide reaction also called the EOB reaction, see Gaspár and Showalter (1990). Using a singular perturbation approach the reaction equations are reduced to (see e.g. [7], chapter 7)

\[
\begin{align*}
\frac{dX}{dt} &= k_1 A_s Y - (k_{-1} + k_2 + k_3 Z_s + k_0) X, \\
\frac{dY}{dt} &= -k_1 A_s Y + (k_{-1} + k_2 + 3k_3 Z_s) X - 2k_3 Y^2 + k_0 (Y_0 - y).
\end{align*}
\]

(1.31)

where the \(k's\) are constants, \(A_s\) and \(Z_s\) are functions of \(X\) and \(Y\) given by

\[
A_s = \frac{k_{-1} X + k_0 A_0}{k_1 Y + k_0}, \quad Z_s = \frac{k_3 Y^2}{k_1 X + k_5 + k_0},
\]

and \(k_0, A_0, Y_0\) are constants. This is an example of a canard system, that is an oscillatory system which undergo sudden major changes in the amplitude and period of the oscillator solution for certain range of the parameter (see [7]).

1.3.4 Schnakenberg

A two-species trimolecular reaction. A trimolecular reaction is when three molecules come together simultaneously to form a product. The following reaction mechanism
called the Schnakenberg (1979) reaction is an example of a trimolecular reaction, it is written as (see e.g. [7], chapter 7)

\[ X \xrightarrow{k_1} A, \quad B \xrightarrow{k_2} Y, \quad 2X + Y \xrightarrow{k_3} 3X. \]

Assuming the concentrations for \( A \) and \( B \) to be constant, the reaction equations are written in nondimensional form as

\[ \frac{du}{dt} = a - u + u^2v, \quad \frac{dv}{dt} = b - u^2v, \]

(1.32)

where \( a, b \) are positive constants. The system (1.32) admits periodic solutions (see [7]).

### 1.3.5 Hodgkin-Huxley, FitzHugh-Nagumo

A neuron is a nerve cell and the axon is part of a neuron. The axon is a long cylindrical tube (like a cable). As part of nerve communication electrical signals propagate along the outer membrane of the axon, this is called an action potential. Here is the Hodgkin-Huxley (1952) model on nerve action potential. In this case the model is in a space-clamped situation, that is without any spatial variation. The equation for the transmembrane potential \( V \) is given by (see e.g. [7], chapter 7)

\[ C \frac{dV}{dt} = -g_{Na}m^3h(V - V_{Na}) - g_Kn^4(V - V_K) - g_L(V - V_L) + I_a, \]

(1.33)

where \( g_{Na}, g_K, g_L, V_{Na}, V_K, V_L, C \) are constants. \( I_a \) is the applied current, supposed to be constant. The variables \( m, n, h \) are bounded by 0 and 1, and defined by the equations (see e.g. [7], chapter 7)

\[ \frac{dm}{dt} = \alpha_m(V)[1 - m] - \beta_m(V)m, \]

\[ \frac{dn}{dt} = \alpha_n(V)[1 - n] - \beta_n(V)n, \]

\[ \frac{dh}{dt} = \alpha_h(V)[1 - h] - \beta_h(V)h, \]

(1.34)

where \( \alpha_m, \alpha_n, \alpha_h \) and \( \beta_m, \beta_n, \beta_h \) are arbitrary functions of \( V \), for specific forms see Keener and Sneyd (1998).

The FitzHugh-Nagumo (FHN) model is not so complex as the Hodgkin-Huxley model. But the FHN model captures the essential behavior of the Hodgkin-Huxley model. The system is written as (see e.g. [7], chapter 7)

\[ \frac{dv}{dt} = v(a - v)(v - 1) - w + I_a, \quad \frac{dw}{dt} = bw - cw, \]

(1.35)

where \( 0 < a < 1, b, \gamma \) are positive constants.
1.3.6 Model for testosterone production

A model for the production of testosterone in the male. According to observations the blood level of testosterone in men should oscillate in time (see [7]). Apart from the concentration of testosterone \( T(t) \), it also involves the concentration of luteinising hormone (LH) \( L(t) \) and LH releasing hormone (LHRH) \( R(t) \). The equations are written (see e.g. [7], chapter 7)

\[
\frac{dR}{dt} = f(T) - b_1 R, \\
\frac{dL}{dt} = g_1 R - b_2 L, \\
\frac{dT}{dt} = g_2 L - b_3 T,
\]

where \( b_1, b_2, b_3, g_1, g_2 \) are positive constants and \( f(T) \) is supposed to be a positive monotonic decreasing function, like e.g. \( f(T) = A / (K + T^m) \). The system (1.36) is similar to the general feedback system (1.30). Here there is a sequence of activation, \( R \) activates \( L \) and \( L \) activates \( T \). \( R, L, T \) are removed from the bloodstream with first-order kinetics. There is also a nonlinear negative feedback by \( T(t) \) on \( R(t) \) via the general feedback function \( f(T) \). A similar model but with delay equations are given in (3.1).

1.4 Belousov-Zhabotinskii reaction

The Belousov-Zhabotinskii (BZ) reaction is an important oscillating chemical reaction. It was first discovered by Boris Belousov (1951) and the work was then continued by Zhabotinskii (1964). The term BZ reaction now refers to a general class of chemical reactions with periodic behavior. There are many reactions involved in the BZ reaction, but it can be modelled by the FKN (Field-Krs-Noyes) model.

1.4.1 Field-Krs-Noyes model

The FKN model contains five reactions which captures the essential elements of the BZ reaction. The reaction system for the five reactions in the FKN model is given by (see e.g. [7], chapter 8)

\[
A + Y \xrightarrow{k_1} X + P, \quad X + Y \xrightarrow{k_2} 2P \\
A + X \xrightarrow{k_3} 2X + 2Z, \quad 2X \xrightarrow{k_4} A + P, \quad Z \xrightarrow{k_5} fY
\]

where \( k_1, \ldots, k_5 \) are rate constants and \( f \) is a stoichiometric factor. Assuming the concentration for \( A \) to be constant and with lowercase letters denoting the concentrations.
Then by applying the law of mass action to (1.37) yields the equations

\[
\begin{align*}
\frac{dx}{dt} &= k_1 ay - k_2 xy + k_3 ax - 2k_4 x^2, \\
\frac{dy}{dt} &= -k_1 ay - k_2 xy + f k_5 z, \\
\frac{dz}{dt} &= 2k_3 ax - k_5 z, \\
\frac{dp}{dt} &= k_1 ay + 2k_2 xy + k_4 x^2,
\end{align*}
\] (1.38)

where \(k_1, \ldots, k_5\) are constants.

1.4.2 Relaxation oscillator

In relaxation oscillators parts of the limit cycle is traversed quickly in comparison with other parts. The FKN system (1.38) has this property. A simple example of a relaxation oscillator is given by (see e.g. [7], chapter 8)

\[
\begin{align*}
\frac{dx}{dt} &= y - f(x), \\
\frac{dy}{dt} &= -x,
\end{align*}
\] (1.39)

where \(0 < \varepsilon \ll 1\) is a constant and \(f(x)\) is a continuous function. For instance with \(f(x) = (1/3)x^3 - x\) we have the Van der Pol oscillator.

1.5 Dynamics of Infectious Diseases

1.5.1 SIR model

In this simple epidemic model, the total population is divided into the distinct classes: \(S\) (susceptibles) can catch disease, \(I\) (infectives) have the disease and can transmit it, \(R\) (removed) can’t catch the disease (immune, dead). The assumptions for the model are that: (i) every pair of individuals has equal probability of coming into contact with one another, (ii) that it is possible to develop immunity after infection, (iii) that there is no incubation period for the disease. With \(S(t), I(t), R(t)\) denoting the population for each class, the system is written as (see e.g. [7], chapter 10)

\[
\begin{align*}
\frac{dS}{dt} &= -rSI, \\
\frac{dI}{dt} &= rSI - aI, \\
\frac{dR}{dt} &= aI,
\end{align*}
\] (1.40)

where \(r, a\) are positive constants. This model is called a SIR model, it was introduced by Kermack-McKendrick (1927). Change between the classes can only be made in the
following way $S \rightarrow I \rightarrow R$. Possible interpretation of the system (1.40). $rSI$ is the transition rate from $S$ to $I$ and $rS$ is the number of susceptibles that catch the disease from each infective, $r$ is a measure of the transmission efficiency of the disease. With no incubation period there is a direct transition from class $S$ to $I$. The term $aI$ is the rate of transition from $I$ to $R$ it could be because of death or recover. By adding the three equations in (1.40) and then integrating we have

\[
S(t) + I(t) + R(t) = N
\]

where $N$ a constant would be the size of the total population. Typical initial conditions are given by

\[
S(0) = S_0 > 0, \quad I(0) = I_0 > 0, \quad R(0) = 0.
\]

### 1.5.2 SI criss-cross model

Sexually transmitted diseases have some difference to other infections such as: its restricted to the sexually active community, may not show immediate symptoms and little or no immunity after infection. Here is a model for venereal diseases. the population is divided into male and females and then for each class it is again divided into a susceptible $S$ and infective class $I$. Male and female classes are distinguish with a bar notation. One assumption for the model are that there is a uniformly promiscuous behavior. The equations are written as (see e.g. [7], chapter 10)

\[
\frac{dS}{dt} = -rSI + aI, \quad \frac{dI}{dt} = rSI - aI, \\
\frac{d\bar{S}}{dt} = -\bar{r}\bar{S}I + \bar{a}\bar{I}, \quad \frac{d\bar{I}}{dt} = \bar{r}\bar{S}I - \bar{a}\bar{I},
\]

(1.41)

where $r, a, \bar{r}, \bar{a}$ are positive constants. This is called a criss-cross $SI$ model. The rate of change from class $S$ to $I$ is given by $rSI$. $r$ is a measure of the transmission efficiency of the disease. After infection one can’t develop immunity, so once healed there is the transition from the $I$ class back to the $S$ class at the rate $aI$. By adding the two equations for males in (1.40) and then also add the two equation for females in (1.40) and then integrating we have

\[
S(t) + I(t) = N, \quad \bar{S}(t) + \bar{I}(t) = \bar{N}.
\]

where $N, \bar{N}$ are constants which are the size of the total population for male and female. Typical initial conditions are given by

\[
S(0) = S_0 > 0, \quad I(0) = I_0 > 0, \quad \bar{S}(0) = \bar{S}_0 > 0, \quad \bar{I}(0) = \bar{I}_0 > 0.
\]
1.5.3 AIDS and HIV model

An epidemic model for AIDS and HIV infection in a homosexual population. Denoting with \( N(t) \) the total population and divide the population into the classes \( X(t) \) susceptibles, \( Y(t) \) infectious, \( A(t) \) people with AIDS and \( Z(t) \) people who developed a non-infectious seropositive form. Assume a constant immigration rate \( B \) into the susceptibles, that for all classes there are a natural death rate \( \mu \), AIDS patients die as cause of the disease at a rate \( d \) and that there is a uniform mixing of the population. 

The equations are written (see e.g. [7], chapter 10)

\[
\frac{dX}{dt} = B - \mu X - \lambda cX, \quad \lambda = \frac{\beta Y}{N}, \\
\frac{dY}{dt} = \lambda cX - (v + \mu)Y, \\
\frac{dA}{dt} = pvY - (d + \mu)A, \\
\frac{dZ}{dt} = (1 - p)vY - \mu Z, 
\]

where \( c, p, d, v, B, \mu, \lambda \) are constants. This could be described as a modified SIR model where the \( R \) class is divided into the classes \( A \) and \( Z \). For infectives \( p \) is the probability to go to the \( A \) class and \((1 - p)\) is the probability to got to the \( Z \) class.

1.5.4 HIV model, combination drug therapy

A model for combination drug therapy. HIV primarily infects a class of white blood cells called CD-4 T-cells. One type of drug is protease inhibitors, they will make newly produced viruses noninfectious. One way to extend the time it takes for the virus to mutate into a drug resistant form is to use a combination of two drugs. Here another kind of drug is also used its called reverse transcriptase inhibitors. The model includes the four species, \( T \) uninfected T-cells, \( T^* \) productively infected T-cells, \( V_I \) infectious viruses, \( V_{NI} \) noninfectious viruses. The equations are given by (see e.g. [7], chapter 10)

\[
\frac{dT}{dt} = s + pT \left( 1 - \frac{T}{T_{\text{max}}} \right) - dT T - kV_I T, \\
\frac{dT^*}{dt} = (1 - n_{rt})kV_I T - \delta T^*, \\
\frac{dV_I}{dt} = (1 - n_p)N\delta T^* - cV_I, \\
\frac{dV_{NI}}{dt} = n_p N \delta T^* - cV_{NI},
\]

(1.43)
where \( k, p, s, d_T, T_{\text{MAX}}, \delta, c, N \) are positive constants and \( 0 \leq n_p \leq 1, 0 \leq n_{rt} \leq 1 \). \( n_p \) is a measure of the effectiveness of a protease inhibitor drug and \( n_{rt} \) is a measure of the effectiveness of a reverse transcriptase inhibitor drug.

### 1.5.5 Immunological response against parasites

A model for the immunological response by the host against gastrointestinal parasites, proposed by Berding et al (1986). Here the model is for an experiment where the host is infected with a constant infection rate. Denote by \( L(t) \) the mean number of larvae per host and by \( M(t) \) the mean number of adult worms per host. The equations are written (see e.g. [7], chapter 10)

\[
\frac{dL}{dt} = \lambda - \mu DL, \\
\frac{dM}{dt} = \mu L - (\delta + I)M, \\
I = \frac{\alpha E^2}{\beta + E^2}, \\
E = \int_{t-T}^{t} L(t') \, dt',
\]

where \( D, \mu, \delta, \lambda \) are constants, \( \delta \) is the natural death rate of the adult worm and \( \lambda \) is the larval infection rate. \( I \) represents the mortality caused by the host’s immune response. Larvae develop to adult worms at a rate of \( \mu L \), but are removed at a rate of \( D \mu L \).

## 2 Models formulated by partial differential equations

### 2.1 General equations, conservation equation

Consider an arbitrary volume \( V \) enclosed by a surface \( S \). A general conservation equation for material in \( V \) can be described as: The rate of change of material in \( V \) equals the material passing through \( S \) plus material created in \( V \). Mathematically this can be written as the equation (see e.g. [7] chapter 11)

\[
\frac{\partial}{\partial t} \int_V c(x,t) \, dv = - \int_S \mathbf{J} \cdot ds + \int_V f \, dv
\]

(2.1)

where \( c(x,t) \) denotes the density of material at point \( x \) in space at time \( t \), \( \mathbf{J}(x,t,c) \) is the flux of material and \( f(x,t,c) \) is the source of material. By using the divergence theorem on equation (2.1) it is possible to derive the equation (see e.g. [7] chapter 11)

\[
\frac{\partial c}{\partial t} + \nabla \cdot \mathbf{J} = f(x,t,c).
\]

(2.2)

This equation is called a conservation equation for \( c \) and will often be used in models later on. Then \( c(x,t) \) could for example be population density or concentration of a
Mathematical models in biology from point of view of Lie group analysis

chemical. In the case of several material we have an equation for a n-dimensional vector \( u \), this is written as

\[
\frac{\partial u}{\partial t} + \nabla \cdot J = f(x, t, u) \tag{2.3}
\]

where now the source \( f(x, t, u) \) is a vector and the flux \( J \) is a tensor.

### 2.1.1 Diffusion

If diffusion is the process by which the material is spread then the flux can be set to

\( J = -D \nabla c \) where the diffusion coefficient \( D \) may be a function of \( x \) and \( c \). Now equation (2.2) becomes (see e.g. [7], chapter 11)

\[
\frac{\partial c}{\partial t} = f(x, t, c) + \nabla \cdot (D \nabla c). \tag{2.4}
\]

This type of equation is refered to as a reaction-diffusion equation. If there are \( n \) number of materials then they can be described by a \( n \)-dimensional vector \( u \) and the equation for \( u \) is given by

\[
\frac{\partial u}{\partial t} = f(x, t, u) + \nabla \cdot (D \nabla u), \tag{2.5}
\]

where \( D \) is a matrix of diffusion coefficients.

### 2.1.2 Chemotaxis

If movement is not random but instead is directed towards some preferred direction, this could for instance be directed movement towards regions with high concentration of a chemical. In this case the chemical is called an attractant and the process controlling the movement is called chemotaxis.

As an example of the chemotaxis process. Lets assume that some cells denoted by \( n(x, t) \) move towards high concentration of an attractant denoted by \( a(x, t) \). Assuming that the cells move up the gradient of the attractant. Then the chemotaxis flux can be written as

\[
J = n \chi(a) \nabla a, \tag{2.6}
\]

where \( \chi(a) \) is a function. Also assuming the cells disperse with diffusion we can write the flux for the cells as

\[
J = J_{\text{diffusion}} + J_{\text{chemotaxis}}
\]

where \( J_{\text{diffusion}} = -D \nabla n \) as in (2.4). Inserting this into the equation (2.2) gives the following, called a reaction-diffusion-chemotaxis equation (see e.g. [7], chapter 11)

\[
\frac{\partial n}{\partial t} = f(n, a) - \nabla \cdot [n \chi(a) \nabla a] + \nabla \cdot (D \nabla n), \tag{2.7}
\]
where $D$ is the diffusion coefficient and $f(n, a)$ is a function representing the cell dynamics. Also we need an equation for the attractant, assuming it disperse by diffusion only we have from equation (2.2) the equation (see e.g. [7], chapter 11)

$$\frac{\partial a}{\partial t} = g(a, n) + \nabla \cdot (D_a \nabla a),$$

(2.8)

where $D_a$ is the diffusion coefficient and $g(a, n)$ is a function representing the kinetics for the attractant.

### 2.1.3 Long range diffusion

Long range diffusion is a type of diffusion more suitable when the density of cells is not small and the short range diffusion is not accurate enough. The flux for long range diffusion can be written as (see e.g. [7] chapter 11)

$$J = D_1 \nabla n - \nabla D_2 \nabla^2 n$$

From the equation (2.2) with $f(n) = 0$ we have

$$\frac{\partial n}{\partial t} = \nabla \cdot D_1 \nabla n - \nabla \cdot (D_2 \nabla^2 n),$$

(2.9)

where $D_1, D_2$ are constants.

### 2.2 Population models

#### 2.2.1 Population model with age distribution

A population model with age distribution. Here $n(t, a)$ is the population density at time $t$ in the age range $a$ to $a + da$. For $t > 0$ and $a > 0$ the governing equation for $n(t)$ is given by (see e.g. [7], chapter 1)

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial a} = \mu(a) n, \quad t > 0, a > 0,$$

(2.10)

where $\mu(a)$ is an arbitrary function which represents the death rate. Appropriate initial conditions are $n(0, a) = f(a)$ and the boundary condition given by

$$n(t, 0) = \int_0^\infty b(a) n(t, a) \, da.$$  

where $b(a)$ is a function representing the birth rate. Equation (2.10) is known as the von Foerster equation.
2.2.2 Insect dispersal

A model for insect dispersal. Assume that insects disperse by diffusion and as the population increase then also the diffusion rate increase. With $n(x,t)$ as population density then the flux is written with density dependent diffusion as (see [7])

$$J = -D(n) \nabla n,$$

(2.11)

here $dD/dn > 0$ reflecting that diffusion increase due to population pressure. Suggested form for the density dependent diffusion coefficient is $D(n) = D_0(n/n_0)^m$, where $m > 0$ and $D_0, n_0$ are positive constants. The equation for the insect population $n$ is obtained from (2.4) and is written as (see e.g. [7], chapter 11)

$$\frac{\partial n}{\partial t} = D_0 \nabla \cdot \left[ \left( \frac{n}{n_0} \right)^m \nabla n \right] + f(n)$$

(2.12)

Possible forms for the growth rate $f(n)$ could be taken from the section on populations models.

2.2.3 Predator-prey model 1

A predator-prey model, here $U(x,t)$ and $V(x,t)$ denote prey respectively predator population density. Both the prey and predator are assumed to spread by diffusion. The population dynamics is similar to the system (1.6) but with logistic growth of the prey. The equations are obtained from (2.5) and are written as (see e.g. [8], chapter 1)

$$\frac{\partial U}{\partial t} = AU \left( 1 - \frac{U}{K} \right) - BUV + D_U \nabla^2 U,$$

$$\frac{\partial V}{\partial t} = CUV - DV + D_V \nabla^2 V,$$

(2.13)

where $D_1, D_2, A, B, C, D, K$ are positive constants. This system is shown in [8] to admit travelling wavefront solutions and having waves of pursuit and evasion, which could be viewed as the predator move to catch the prey and the prey move to evade from the predators.

2.2.4 Predator-prey model 2

A one-dimensional predator-prey model with $u(x,t)$ denoting the prey and $v(x,t)$ the predator density. Suppose that the prey move at the speed $c_1$ and the predators at the speed $c_2$. If the prey and predators come in contact then the prey try to evade the predators while the predators start pursuing the prey. This motivates the following two flux terms for the prey and predator (see e.g. [8], chapter 1)

$$J_u = -(c_1 + h_1 v_x) u, \quad J_v = -(c_2 - h_2 u_x) v.$$
Then the equations for $u$ and $v$ are obtained via the equations (2.3) and are written as

$$
\begin{align*}
\frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left[ (c_1 + h_1 \frac{\partial v}{\partial x}) u \right] &= f(u, v), \\
\frac{\partial v}{\partial t} - \frac{\partial}{\partial x} \left[ (c_2 - h_2 \frac{\partial u}{\partial x}) v \right] &= g(u, v)
\end{align*}
$$

(2.14)

where $c_1, c_2, h_1, h_2$ are constants, $f(u, v), g(u, v)$ are functions representing the population dynamics.

### 2.2.5 Competition between grey and red squirrels

Grey squirrels were introduced in Britain where there already existed red squirrels. Here is a competition model where the red and grey squirrels are assumed to compete for the same food and both disperse by diffusion. Denoting by $S_1(X, T)$ and $S_2(X, T)$ the population density for the grey respectively the red squirrel. The equations are written as (see e.g. [8], chapter 1)

$$
\begin{align*}
\frac{\partial S_1}{\partial T} &= D_1 \nabla^2 S_1 + a_1 S_1 (1 - b_1 S_1 - c_1 S_2), \\
\frac{\partial S_2}{\partial T} &= D_2 \nabla^2 S_2 + a_2 S_2 (1 - b_2 S_2 - c_2 S_1)
\end{align*}
$$

(2.15)

where $a_1, a_2, b_1, b_2, c_1, c_2, D_1, D_2$ are non-negative constants.

### 2.2.6 Competition model

A model for competition between genetically engineered microbes denoted by $E(x, t)$ and natural (unmodified) microbes denoted by $N(x, t)$. The equations are written as (see e.g. [8], chapter 1)

$$
\begin{align*}
\frac{\partial E}{\partial t} &= \frac{\partial}{\partial x} \left( D(x) \frac{\partial E}{\partial x} \right) + r_E E [G(x) - a_E E - b_E N], \\
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial x} \left( d(x) \frac{\partial N}{\partial x} \right) + r_N N [g(x) - a_N N - b_N E]
\end{align*}
$$

(2.16)

where $r_E, r_N, a_E, a_N, b_E, b_N$ are constants. $d(x), D(x)$ are space dependent diffusion coefficients and $g(x), G(x)$ quantify the carrying capacities. The functions $d(x), D(x), g(x), G(x)$ could be set to periodic functions. In this way it is possible to model a spatially heterogeneous environment, see [8].
2.2.7 Interaction between two wolf packs

A model for two wolf packs, from the book [8], chapter 14. The wolves uses raised leg urination (RLU) to mark their territory. The location of the wolves is described by a probability density function. The expected density is the probability of finding a wolf at point \( x \) and time \( t \). The expected density of wolves in pack 1 is denoted by \( u(x, t) \) and in pack 2 by \( v(x, t) \). In this model only wolves that can leave RLU is considered (a few in each pack). Also needed is the expected density of the RLU from each pack, RLU left from pack 1 is denoted \( p(x, t) \) and for pack 2 by \( q(x, t) \). The point \( x_u \) denotes the location of the den for pack 1 and \( x_v \) the location of the den for pack 2.

The wolves are assumed to search for food randomly and then to move directly back towards the den. Also if the wolves come in contact with RLU from another wolf pack then the wolves move away from the foreign RLU in the direction towards their den. The direct movement of the wolves back to their den is represented by the flux \( J_{cu} \) for pack 1 and \( J_{cv} \) for pack 2

\[
J_{cu} = -uc_u(x - x_u, q), \quad J_{cv} = -vc_v(x - x_v, p).
\] (2.17)

The functions \( c_u(x - x_u, q) \) and \( c_v(x - x_v, p) \) are suggested to be bounded monotonically increasing functions of \( q \) respectively \( p \). The random movement in search for food is represented by the flux \( J_{du} \) for pack 1 and \( J_{dv} \) for pack 2, they are given by

\[
J_{du} = -d_u(u)\nabla u, \quad J_{dv} = -d_v(v)\nabla v
\] (2.18)

where \( dd_u/du \geq 0 \) and \( dd_v/dv \geq 0 \). The movement away from other wolf packs RLU is represented by the flux \( J_{au} \) for pack 1 and \( J_{av} \) for pack 2, they are given by

\[
J_{au} = a_u(q)u\nabla q, \quad J_{av} = a_v(p)v\nabla p
\] (2.19)

where \( da_u/dq \geq 0 \) and \( da_v/dp \geq 0 \).

The equations for the wolf packs and the RLU markings are written as (see e.g. [8], chapter 14)

\[
\begin{align*}
\frac{\partial u}{\partial t} + \nabla \cdot [J_{cu} + J_{du} + J_{au}] &= 0, \\
\frac{\partial v}{\partial t} + \nabla \cdot [J_{cv} + J_{dv} + J_{av}] &= 0, \\
\frac{\partial p}{\partial t} &= u[l_p + m_p(q)] - f_p p, \\
\frac{\partial q}{\partial t} &= v[l_q + m_q(p)] - f_q q,
\end{align*}
\] (2.20)

where \( l_p, l_q, f_p, f_q \) are constants. The functions \( m_p(q) \) and \( m_q(p) \) are typically bounded and monotonically nondecreasing functions, such as for example

\[
m_p(q) = \frac{Aq}{B + q}, \quad m_q(p) = \frac{Cp}{D + p}
\]
where $A, B, C, D$ are constants.

The system (2.20) could also be considered in the case with $J_{a_u} = 0, J_{a_v} = 0$.

### 2.2.8 Interaction between two wolf packs and deer

A model of the interaction between two wolf packs and a deer population. $u, v, p, q$ have the same meaning as in the model in section 2.2.7. The expected density of the deer is denoted by $h(x, t)$. In this model wolf movement is not effected by RLU marking, movement is only considered as searching for food and going back to the den. In the search for food, the wolfs are assumed to move towards the regions where the deer is (and not as before randomly). This is represented by the flux $J_{d_u}$ for the wolf pack 1 and $J_{d_v}$ for the wolf pack 2, they are given by

$$
J_{d_u} = \sigma_u u \nabla h, \quad J_{d_v} = \sigma_v v \nabla h
$$

The deer population is assumed to have no motion, no growth and is only considered to change due to predation by the wolfs. The equations are written (see e.g. [8], chapter 14)

$$
\frac{\partial u}{\partial t} = \nabla \cdot \left[ c_u (x - x_u) u - \sigma_u u \nabla h \right], \\
\frac{\partial v}{\partial t} = \nabla \cdot \left[ c_v (x - x_v) v - \sigma_v v \nabla h \right], \\
\frac{\partial p}{\partial t} = u[l_p + m_p(q, h)] - f_p p, \\
\frac{\partial q}{\partial t} = v[l_q + m_q(p, h)] - f_q q, \\
\frac{\partial h}{\partial t} = - (\alpha_u u + \alpha_v v) g(h).
$$

(2.21)

where $\sigma_u, \sigma_v, l_p, l_q, f_p, f_q, \alpha_u, \alpha_v$ are constants. Suggested forms for $g(h)$ is a saturating function such as

$$
g(h) = \frac{ah^m}{1 + bh^m}
$$

where $a, b$ are constants and $m \geq 1$.

### 2.3 Pattern formation with reaction diffusion

#### 2.3.1 Introduction

Here in this section the models are for spatial pattern formation. Wave phenomena create spatial pattern, but these are spatio-temporal patterns. Here it is the formation of steady state spatially heterogeneous spatial patterns, [8]. One area where this is studied is for the spatial pattern formation in embryology. The part of embryology that is
concerned with the development of pattern and form in embryos is called morphogenesis. The proposed pattern formation process for morphogenesis with reaction diffusion models is the following. With the interaction of reaction and diffusion the chemicals (here called morphogens) can under certain conditions form a pattern (pre-pattern). Also it is assumed that there is a uniform density of cells. The cells are assumed to be able to sense the chemical concentration levels and if the level is above a certain threshold then the cells differentiate. The basic ideas for using reaction diffusion models for morphogenesis is from Turing (1952).

2.3.2 General reaction diffusion systems

For a two chemical reaction diffusion mechanism with \( U(X, T), V(X, T) \) denoting the chemical concentrations, the equations are given by (see equation (2.5))

\[
\frac{\partial U}{\partial T} = F(U, V) + D_U \nabla^2 U,
\]

\[
\frac{\partial V}{\partial T} = G(U, V) + D_V \nabla^2 V,
\]

where \( D_U, D_V \) are the diffusion coefficients and the two functions \( F(U, V), G(U, V) \) represents the reaction kinetics.

For specific \( F(U, V), G(U, V) \) the system (2.22) is usually written in nondimensional form as (see e.g. [8], chapter 2)

\[
\begin{align*}
    u_t &= \gamma f(u, v) + \nabla^2 u \\
    v_t &= \gamma g(u, v) + d \nabla^2 v
\end{align*}
\]

(2.23)

where \( \gamma, d \) are constants. As an example of how to transform from the system (2.22) to the nondimensional form (2.23) consider the reaction diffusion system

\[
\begin{align*}
    \frac{\partial U}{\partial T} &= k_1 - k_2 U + \frac{k_3 U^2}{V} + D_U \nabla^2 U, \\
    \frac{\partial V}{\partial T} &= k_4 U^2 - k_3 V + D_V \nabla^2 V,
\end{align*}
\]

(2.24)

which is of the form (2.22). By using the transformations

\[
\begin{align*}
    u &= \frac{k_4}{k_3} U, & v &= \frac{k_4 k_5}{k_3^2} V, & t &= D_U T, & x &= X, \\
    a &= \frac{k_1 k_4}{k_3 k_5}, & b &= \frac{k_2}{k_5}, & \gamma &= \frac{k_5}{D_U}, & d &= \frac{D_V}{D_U}
\end{align*}
\]

we find a system of the form (2.23), this system is written as

\[
\begin{align*}
    u_t &= \gamma \left( a - bu + \frac{u^2}{v} \right) + \nabla^2 u, \\
    v_t &= \gamma (u^2 - v) + d \nabla^2 v.
\end{align*}
\]

(2.25)
Some possible kinetic functions \( f(u, v) \), \( g(u, v) \) for (2.23) are here given by

\[
\begin{align*}
  f(u, v) &= a - u + u^2 v, \quad g(u, v) = b - u^2 v, \\
  f(u, v) &= a - u - \frac{\rho uv}{1 + u + Ku^2}, \quad g(u, v) = \alpha(b - v) - \frac{\rho uv}{1 + u + Ku^2}, \\
  f(u, v) &= a - bu + \frac{u^2}{v(1 + ku^2)}, \quad g(u, v) = u^2 - v, \\
  f(u, v) &= a - bu + \frac{u^2}{v}, \quad g(u, v) = u^2 - v, \\
  f(u, v) &= \frac{u^2}{v} - bu, \quad g(u, v) = u^2 - v.
\end{align*}
\]

where \( a, b, \rho, K, \alpha \) are constants. The first three reaction kinetics are from Schnakenberg reaction (see (1.32)), Thomas reaction (see (1.26)), Gierer and Meinhardt reaction (see (1.27)). Suitable initial conditions and zero flux boundary condition for (2.23) could be written as (see e.g. [8])

\[
u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad (n \cdot \nabla) \begin{pmatrix} u \\ v \end{pmatrix} = 0, \text{ for } x \text{ on } \partial B
\]

for a domain \( B \), where \( \partial B \) is the closed boundary and \( n \) the unit outward normal to \( \partial B \).

If in the absence of diffusion in (2.23), the homogeneous steady state is stable to small perturbations and unstable when diffusion is present then the system is said to exhibit diffusion-driven instability. And may evolve to a bounded, stationary, spatially nonhomogeneous, steady state, that is a spatial pattern. The homogeneous steady state \((u_0, v_0)\) to (2.23) without diffusion are solutions of \( f(u, v) = 0, g(u, v) = 0 \), this steady state must be linearly stable. Then the steady state must be unstable when (2.23) is linearised about \((u_0, v_0)\).

The conditions for diffusion-driven instability for the two-species reaction diffusion system (2.23) are given by (see e.g. [8], chapter 2)

\[
\begin{align*}
  f_u + g_v &< 0, \quad f_u g_v - f_v g_u > 0, \quad df_u + g_v > 0, \\
  (df_u + g_v)^2 &- 4d(f u g_v - f_v g_u) > 0.
\end{align*}
\]

(2.26)

here \( f_u, f_v, g_u, g_v \) are evaluated at the steady state \((u_0, v_0)\). The conditions (2.26) leads to \( d \neq 1 \).

2.3.3 Population model 1

Looking at a reaction diffusion equation from an ecological point of view. Here a population model with \( u(x, t) \) as the population density. The equation is written as (see e.g. [8], chapter 2)

\[
u_t = f(u) + D \nabla^2 u.
\]

(2.27)
The population disperse by diffusion with diffusion coefficient $D$ and $f(u)$ represents the population dynamics. A possible form for $f(u)$ could be

$$f(u) = ru \left(1 - \frac{u}{q}\right) - \frac{bu^2}{a^2 + u^2}$$

where $r, q, a, b$ are constants, this is from the model for the spruce budworm (see equation (1.5)). In the one dimensional case with suitable living condition for the specie within the range $[0, L]$ and unsuitable living conditions outside this range. Then appropriate initial and boundary conditions are given by (see [8])

$$u(x, 0) = u_0(x), \quad u(0, t) = u(L, t) = 0.$$  

**2.3.4 Population model 2**

With $n(x,t)$ as population density, logistic population growth and assuming population disperse by diffusion. Then from (2.2) with the source term $f(n) = rn(1 - n/K)$ and flux $J = D\nabla n$, we have the equation (see e.g. [7], chapter 11)

$$\frac{\partial n}{\partial t} = rn\left(1 - \frac{n}{K}\right) + D\nabla^2 n, \quad (2.28)$$

where $r, K, D$ are constants. This equations is called Fisher-Kolmogoroff equation.

**2.3.5 Butterfly wing patterns**

A model for patterns on butterfly wings, especially a pattern type called central symmetry pattern. Let the wing be defined as a sector with angle $\theta$ and radii $r$, the range for the angle is $0 \leq \theta \leq \theta_0$ and for the radii $r_1 \leq r \leq r_2$. At a ceratin level of a morphogen concentration the cells on the wing are assumed to react and a gene $G$ is activated to produce a product $g$. The morphogen is assumed to disperse by diffusion and degrade according to first-order kinetics. $S(r, \theta, t)$ denotes the concentration of the morphogen and $g(t; r, \theta)$ the gene product. $g(t; r, \theta)$ depend on $\theta, r$ only through $S$.

The equations for $S$ and $g$ are written in nondimensional form as (see e.g. [8], chapter 3)

$$\frac{\partial S}{\partial t} = \frac{\partial^2 S}{\partial r^2} + \frac{1}{r} \frac{\partial S}{\partial r} + \frac{1}{r^2} \frac{\partial^2 S}{\partial \theta^2} - \gamma k S,$$

$$\frac{dg}{dt} = \gamma \left(k_1 S + \frac{k_2 g^2}{1 + g^2} - k_3 g\right), \quad (2.29)$$

where $k, k_1, k_2, k_3, \gamma$ are constants. The equation for $S$ is a reaction diffusion equation in polar coordinates. The morphogen is assumed to be released at the anterior $(r_A, \theta_0)$ and posterior $(r_P, 0)$ of the wing, an with zero flux for the morphogen at the
boundaries. Then the initial and boundary conditions are given by (see e.g. [8], chapter 3)

\[
S(r, \theta, 0) = 0, \quad r_1 < r < r_2, \quad 0 < \theta < \theta_0
\]

(2.30)

\[
g(0; r, \theta) = 0, \quad 0 \leq \theta \leq \theta_0, \quad r = r_1, \quad r = r_2,
\]

(2.31)

where \( \delta(t) \) is the Dirac delta function.

\section{2.4 Bacterial patterns}

\subsection{2.4.1 Bacterial pattern 1}

A model for bacterial pattern formation. The model involves the bacteria E. coli and S. typhimurium in a semi-solid medium. The bacteria is assumed to be able to produce a chemical and the bacteria will move preferentially towards high concentrations of this chemical (chemoattractant). The bacteria consumes a stimulant (nutrient). The bacteria, chemoattractant and stimulant is assumed to disperse by diffusion. The bacterial cell density, the concentration of the chemoattractant and the concentration of the stimulant are denoted by \( n(x; t) \), \( c(x; t) \) and \( s(x; t) \). The equations are obtained from the basic reaction diffusion equation (2.4) and chemotaxis equation (2.7) and are written as (see e.g [8], chapter 5)

\[
\frac{\partial n}{\partial t} = D_n \nabla^2 n - \nabla \cdot [\chi(n, c) \nabla c] + k_3 n \left( \frac{k_4 s^2}{k_9 + s^2} - n \right),
\]

\[
\frac{\partial c}{\partial t} = D_c \nabla^2 c + k_5 s \frac{n^2}{k_6 + n^2} - k_7 n c,
\]

\[
\frac{\partial s}{\partial t} = D_s \nabla^2 s - k_8 n \frac{s^2}{k_9 + s^2},
\]

(2.32)

where the \( k \)'s and \( D_n, D_c, D_s \) are constants. Suggested form for the chemotaxis function \( \chi(n, c) \) is \( \chi(n, c) = k_1 n / (k_2 + c)^2 \). This model is for a semi-solid medium, a model suitable for a liquid medium is obtain by setting \( k_3 = 0, k_7 = 0, k_8 = 0 \) (see [8]).

\subsection{2.4.2 Bacterial pattern 2}

A model for patterns created by the bacteria Bacillus subtilis. Let \( n(x; t) \) denote the nutrient concentration and \( b(x; t) \) the bacterial cell density. The equations are written
as (see e.g. [8], chapter 5)

\[
\frac{\partial n}{\partial t} = D_n \nabla^2 n - \frac{knb}{1 + \gamma n},
\]

\[
\frac{\partial b}{\partial t} = \nabla \cdot (D_b \nabla b) + \theta \frac{knb}{1 + \gamma n},
\]

(2.33)

where \(k, \gamma, \theta, D_n\) are constants and the diffusion coefficient \(D_b\) is suggested to take the form \(D_b = \sigma nb\) where \(\sigma\) a constant.

### 2.5 Cell-chemotaxis models

For cell-chemotaxis models there is a different approach to pattern formation than with reaction-diffusion models. Here with actual cell movement, the pattern is in the cell density and the assumption is that cells which are in high-density aggregates then differentiate.

#### 2.5.1 General model with a special case

The cell is assumed to secrete a chemical and also move towards high concentration of the chemical. The cells and the chemical is assumed to disperse by diffusion. Denote by \(n(x, t)\) the cell density and by \(c(x, t)\) the concentration of the chemoattractant. The equations for a general cell chemotaxis model could be written as (see e.g. [9])

\[
\frac{\partial n}{\partial t} = \nabla \cdot (D_n(c)\nabla n) - \nabla \cdot (nf(n, c)\nabla c) + h(n, c),
\]

\[
\frac{\partial c}{\partial t} = \nabla \cdot (D_c \nabla c) + g(n, c),
\]

(2.34)

where \(D_n, D_c\) are the diffusion coefficients. The function \(h(n, c)\) represents the cell division and cell death. The function \(f(n, c)\) represents the chemotactic response of the cells. The function \(g(n, c)\) represents the production and degradation of the chemoattractant. See also the equations (2.7),(2.8).

A special case of the equations (2.34) is given with the functions (see e.g. [9])

\[
h(n, c) = r n \left(1 - \frac{n}{n_0}\right), \quad f(n, c) = \alpha, \quad g(n, c) = \frac{\nu n}{n + \gamma} - \mu c,
\]

where \(r, n_0, \alpha, \nu, \gamma, \mu\) are constants, also the diffusion coefficient \(D_n(c)\) is taken as a constant. Then the equations (2.34) is written as

\[
\frac{\partial n}{\partial t} = \nabla \cdot (D_n \nabla n) - \nabla \cdot (na \nabla c) + r n \left(1 - \frac{n}{n_0}\right),
\]

\[
\frac{\partial c}{\partial t} = \nabla \cdot (D_c \nabla c) + \frac{\nu n}{n + \gamma} - \mu c.
\]

(2.35)
2.5.2 Model for a slime mould

The diffusion-chemotaxis system (2.7),(2.8) can be used as a model for the slime mould \textit{Dictyostelium discoideum}. Under certain conditions the slime mould (a single cell amoebae) move towards high concentrations of a chemical called cyclic-AMP. The slime mould is also able to produce this chemical. Denoting by \( n(x, t) \) the amoeba cell density and by \( a(x, t) \) the concentration of the cyclic-AMP chemical. Setting the source term for the slime mould to \( f(n, a) = 0 \). Also setting the production of the attractant proportional to the amoeba density and death rate proportional to the attractant concentration gives the source term \( g(a, n) = h n - k a \). Then the equations are then given by (see e.g. \cite{7}, chapter 11)

\[
\frac{\partial n}{\partial t} = -\nabla \cdot n \chi(a) \nabla a + \nabla \cdot D \nabla n,
\]

\[
\frac{\partial a}{\partial t} = h n - k a + \nabla \cdot D_a \nabla a,
\]

(2.36)

where \( h, k, D, D_a \) are positive constants. Some suggested forms for \( \chi(a) \) are given by

\[
\chi(a) = \chi_0, \quad \chi(a) = \frac{\chi_0}{a}, \quad \chi(a) = \frac{\chi_0 K}{(K + a)^2},
\]

where \( \chi_0 > 0, K > 0 \).

2.6 Mechanochemical models

Here in this section we deal with the Murray-Oster mechanochemical approach to pattern generation, for more information see \cite{8}, chapter 6. The mechanochemical theory uses a different approach to pattern formation than the reaction diffusion theory. Here the cells themselves form a spatial pattern. The assumption is that high-density aggregates then differentiates. The mechanochemical approach considers the role that mechanical forces play in the process.

A mechanochemical model for early embryonic cells. The early embryonic cells of interest here is called mesenchymal, fibroblast or dermal cells. These cells can move within the extracellular matrix (ECM), the cells secrete fibrous material which helps to build up the ECM. Also the cells can generate large traction forces. Here is a model for pattern formation of such cells. Setting \( n(r, t) \) as cell density, \( \rho(r, t) \) as density of ECM and \( u(r, t) \) as the displacement vector of the ECM. The displacement vector is defined so that a material point in the ECM initially at \( r_0 \) is at time \( t_0 \) at the position \( r_0 + u(r_0, t) \).

The equation for the cell density is of the form of the conservation equation (2.2) and is written as (see e.g. \cite{8}, chapter 6)

\[
\frac{\partial n}{\partial t} + \nabla \cdot J = f(x, t, n)
\]

(2.37)
where $J$ is the flux and $f(x, t, n)$ represents the cell proliferation rate. The function $f(x, t, n)$ is suggested to take a logistic model $f(x, t, n) = rn(N - n)$. Some suggested flux terms are convective $J_c$, long range diffusion $J_D$ and haptotaxis $J_h$, these are written (see e.g. [8], chapter 6)

$$
\begin{align*}
J_c &= n \frac{\partial u}{\partial t}, \\
J_D &= -D_1 \nabla n + D_2 \nabla(\nabla^2 n), \\
J_h &= n(a_1 \nabla \rho - a_2 \nabla^3 \rho),
\end{align*}
$$

(2.38)

where $a_1, a_2, D_1, D_2$ are positive constants. $\frac{\partial u}{\partial t}$ could be described as the velocity at which the matrix deforms.

The composite material of cell and matrix is modelled as a linear, isotropic viscoelastic continuum at mechanical equilibrium for the cells and matrix. The equation for the mechanical interaction between cells and the ECM within which they move, is given by (see e.g. [8], chapter 6)

$$
\nabla \cdot \sigma + \rho F = 0
$$

(2.39)

where $F$ is the external force acting on the matrix and $\sigma$ is the stress tensor. Dividing the the stress tensor into one term for the ECM and one for the cells, it is then written as $\sigma = \sigma_{\text{ECM}} + \sigma_{\text{cell}}$. Suggested forms for $\sigma_{\text{ECM}}$ and $\sigma_{\text{cell}}$ are (see e.g. [8], chapter 6)

$$
\begin{align*}
\sigma_{\text{ECM}} &= (\mu_1 \varepsilon_t + \mu_2 \theta_t) \mathbf{I} + E'(\varepsilon + \nu' \theta) \mathbf{I}, \\
\sigma_{\text{cell}} &= \frac{\tau n}{1 + \lambda n^2} (\rho + \gamma \nabla^2 \rho) \mathbf{I},
\end{align*}
$$

(2.40)

where $\mu_1, \mu_2, \tau, \lambda, \gamma$ are constants, $\theta = \nabla \cdot u$, $E' = E/(1 + \nu)$, $\nu' = \nu/(1 - 2\nu)$. $\theta$ is called dilation, $E$ is the Young’s modulus and $\nu$ the Poisson ratio. $\varepsilon$ is the elastic strain tensor of the ECM and is given by

$$
\varepsilon = \frac{1}{2}(\nabla u + \nabla u^T).
$$

(2.41)

The equation for $\rho(r, t)$, the material in the ECM is taken in the form of the conservation equation (2.2) and is written as (see e.g. [8], chapter 6)

$$
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u_t) = S(n, \rho, u)
$$

(2.42)

where $S(n, \rho, u)$ is the rate of secreted matrix material from the cells.

Putting these three equations together, by taking equation (2.37) with $f(x, t, n) = rn(N - n)$, equation (2.39) with $F = -su$ and equation (2.42) with $S(n, \rho, u) = 0$
and with the flux terms $J_c, J_D, J_h$ from (2.38) and $\sigma_{\text{ECM}}, \sigma_{\text{cell}}$ from (2.40). The equations are then written as
\[
\frac{\partial n}{\partial t} + \nabla \cdot (J_c + J_D + J_h) = r_n(N - n),
\]
\[
\nabla \cdot (\sigma_{\text{ECM}} + \sigma_{\text{cell}}) - \rho s u = 0,
\]
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u_t) = 0.
\]  
(2.43)

Possible extension to the equation (2.37) could be to add some flux like galvanotaxis or chemotaxis, these could take the form (see [8])
\[
J_G = g n \nabla \phi, \quad J_C = c \nabla c
\]  
(2.44)

where $g, \chi$ are positive constants and $\phi$ is an electric potential.

A nondimensional of the equations (2.43) with $S(n, \rho, u) = 0$ yields (see [8], chapter 6)
\[
\frac{\partial n}{\partial t} = D_1 \nabla^2 n - D_2 \nabla^4 n - \nabla \cdot \left[ a_1 n \nabla \rho - a_2 n \nabla (\nabla^2 \rho) \right] - \nabla \cdot (n u_t) + r_n(1 - n),
\]
\[
\nabla \cdot \left[ (\mu_1 \varepsilon_t + \mu_2 \theta_t \mathbf{I}) + (\varepsilon + \nu' \theta \mathbf{I}) + \frac{\tau n}{1 + \lambda n^2} (\rho + \gamma \nabla^2 \rho) \mathbf{I} \right] = s \rho u_t,
\]
\[
\rho + \nabla \cdot (\rho u_t) = 0,
\]  
(2.45)

where $D_1, D_2, a_1, a_2, r, \tau, \lambda, s, \mu_1, \mu_2, \gamma, \nu'$ are positive constants. $\mathbf{I}$ is identity matrix.

Possible simplifications of (2.45) can be obtained by using one or more of the following alternatives: $a_1 = a_2 = 0$ means no haptotaxis, $D_1 = D_2 = 0$ no cell diffusion, $D_2 = 0$ no long range diffusion, $r = 0$ no cell proliferation, $\gamma = 0$ no cell-ECM interaction and $\mu_1 = \mu_2 = 0$ no viscoelastic effect in the ECM.

### 2.6.1 Fingerprint pattern model

A model for dermatoglyphic (fingerprints) pattern formation. It is based on the premiss that the pattern formation is initiated in the dermis and that that the pattern in the epidermis is a consequence of the pattern in the dermis. A version of the equations (2.43) with $D_2 = a_2 = 0$, $\lambda = 0$ and $S(n, \rho, u) = mn\rho(\rho_0 - \rho)$. Is then used as a model for pattern formation in the dermis (see e.g. [8], chapter 6)
\[
\frac{\partial n}{\partial t} = D_1 \nabla^2 n - \nabla \cdot (n u_t) - a_1 \nabla \cdot n \nabla \rho + r_n(N - n),
\]
\[
\nabla \cdot \left[ \mu_1 \varepsilon_t + \mu_2 \theta_t \mathbf{I} + E'(\varepsilon + \nu' \theta \mathbf{I}) + \tau n(\rho + \gamma \nabla^2 \rho) \mathbf{I} \right] - s \rho u = 0,
\]
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u_t) = mn\rho(\rho_0 - \rho).
\]  
(2.46)

where $m, \rho_0$ are constants and also with the same constants as in (2.43).
2.6.2 Model for embryonic epithelium

A mechanochemical model for the epithelium (as part of the epidermis) in embryos. Epithelial cells is an early embryonic cell type. The cells in the epithelium can not actively migrate but are arranged in layers or sheets. The layer can bend and deform. Cytogel is the interior of the cell. Depending on the local level of free calcium, the cytogel can become like sol or gel. This means that the level of free calcium changes the viscosity and elasticity of the cytogel. The epithelial sheet of cells is modelled as a viscoelastic continuum of cytogel, lets denote by $c(r,t)$ the concentration of free calcium and by $u(r,t)$ the displacement vector. The equations are written in nondimensional form as (see e.g. [8], chapter 6)

$$\nabla \cdot [\mu_1 \varepsilon + \mu_2 \delta_{\varepsilon} \mathbf{I} + \varepsilon + \nu' \delta_{\theta} + \tau(c) \mathbf{I}] = s u,$$

$$\frac{\partial c}{\partial t} = D \nabla^2 c + R(c) + \gamma \theta,$$

(2.47)

where $D, \mu_1, \mu_2, \nu', \gamma$ are constants, $\mathbf{I}$ is the identity matrix. The dilation $\theta = \nabla \cdot u$ and $\varepsilon$ the strain tensor is the same as in (2.41) and $\tau(c)$ ia a function. Suggested form for the calcium kinetics $R(c)$ is given by $R(c) = \alpha c^2/(1 + \beta c^2)$.

2.6.3 Cytogel model

Here is a model that try to capture the formation of foldings on the cell membrane, taken as a mechanochemical model for the cytogel sheet. The cytogel is the interior of the cell. The cytogel is taken as a viscoelastic continuum with the two components sol and gel. The reversible transition from sol to gel is regulated by the calcium level. The concentrations for sol, gel and calcium is denoted by $S(x,t), G(x,t)$, and $c(x,t)$ respectively. Assume movement of the sol, gel and calcium can be described with diffusion and for the sol and gel also with convection. In one-dimensional space the equations for the sol, gel and calcium are written as (see e.g. [8], chapter 6)

$$S_t + \frac{\partial}{\partial x} (Su_x) = D_S S_{xx} - F(S,G,\varepsilon),$$

$$G_t + \frac{\partial}{\partial x} (Gu_x) = D_G G_{xx} + F(S,G,\varepsilon),$$

$$c_t = D_c c_{xx} + R(c,\varepsilon),$$

(2.48a)

$$\frac{\partial}{\partial x} \left[ \frac{\pi}{1 + \varepsilon} - G \varepsilon - \beta \varepsilon_{xx} - \frac{G \tau(c)}{1 + \varepsilon^2} - G \mu \varepsilon_t \right] = 0.$$

(2.48b)

where $D_S, D_G, D_c, \pi, E, \beta > 0, \mu$ are constants, and the strain $\varepsilon = u_x$. The function $R(c,\varepsilon)$ represents the calcium kinetics and the function $F(S,G,\varepsilon)$ the sol-gel kinetics. The suggested form [8] $F(S,G,\varepsilon) = k_+(\varepsilon) S - k_-(\varepsilon) G$ where the function $k_+ (\varepsilon)$ increase and $k_- (\varepsilon)$ decrease.
The system (2.48) can be simplified to the system (see e.g. [8], chapter 6)

\[ G_{\varepsilon_t} = G_{\varepsilon_xx} + f(G, \varepsilon), \]
\[ G_t + \frac{\partial}{\partial x}(G u_t) = D G_{xx} + g(G, \varepsilon) \]

(2.49)

where \( D \) is a constant and

\[ f(G, \varepsilon) = -\sigma_0 + \frac{\pi}{1 + \varepsilon} - \frac{G \tau}{1 + \varepsilon^2} - G \varepsilon, \]
\[ g(G, \varepsilon) = k_+ (\varepsilon) - [k_+ (\varepsilon) + k_- (\varepsilon)] G, \]

(2.50)

where \( \tau, \sigma_0 \) are constants.

### 2.6.4 Tissue interaction between the epidermis and the dermis

A tissue interaction model between the epidermis and the dermis. Epidermal and dermal cell density is denoted by \( N(x, t) \) and \( n(x, t) \) respectively. Two signal morphogens, one produced in the epidermis, denoted by \( \varepsilon(x, t) \) and the other is produced in the dermis and is denoted by \( s(x, t) \). The morphogen signal \( \varepsilon \) can diffuse to the dermis, then it is denoted by \( \varepsilon \), and then it functions as a chemoattractant for the dermal cells. The morphogen \( s \) can diffuse to the epidermis, then it is denoted by \( \hat{s} \), and then it cause the epidermal cells to aggregate. Also assume epidermal cells move by convection and dermal by diffusion. The epithelial sheet is taken to be a viscoelastic continuum at equilibrium. \( u(x, t) \) is the displacement of a material point in the epidermis which was initially at \( x \). The epidermis part of the equations are written as (see e.g. [8], chapter 6)

\[ \frac{\partial N}{\partial t} = -\nabla \cdot \left[ N \frac{\partial u}{\partial t} \right], \]
\[ \nabla \cdot \left[ \mu_1 \varepsilon_t + \mu_2 \theta_t \mathbf{I} + E'[(\varepsilon - \beta_1 \nabla^2 \varepsilon) + \nu'(\theta - \beta_2 \nabla^2 \theta) \mathbf{I}] \right] + \tau s^2 (1 + c s^2)^{-1} \mathbf{I} - \rho u = 0, \]
\[ \frac{\partial \hat{\varepsilon}}{\partial t} = \hat{D}_\varepsilon \nabla^2 \hat{\varepsilon} + f(N, \hat{s}) - P_e (\hat{\varepsilon} - \varepsilon) - \gamma \hat{\varepsilon}, \]
\[ \frac{\partial \hat{s}}{\partial t} = \hat{D}_s \nabla^2 \hat{s} + P_s (s - \hat{s}) - \nu N \hat{s}, \]

where \( \hat{D}_\varepsilon, \hat{D}_s, \beta_1, \beta_2, P_e, P_s, \rho, \tau, \gamma, \nu, c \) are constants. The function \( f(N, \hat{s}) \) represents the production of the morphogen \( \hat{\varepsilon} \), it is supposed to increase with \( N \) and decrease with \( \hat{s} \).
The dermis part of the equations are written as (see e.g. [8], chapter 6)

\[
\frac{\partial n}{\partial t} = D \nabla^2 n - \alpha \nabla \cdot n \nabla e + rn(n_0 - n), \\
\frac{\partial s}{\partial t} = D_s \nabla^2 s + g(n, e) - P_s(s - \bar{s}) - \nu N \bar{s}, \\
\frac{\partial e}{\partial t} = D_e \nabla^2 e + P_e(\hat{e} - e) - \gamma n e,
\]

(2.52)

where \( D_e, D_s, \gamma, \nu \) are constants, and \( g(n, e) \) is a function representing the production of the morphogen \( s \), it is supposed to increase with \( n \).

2.7 Vascular network formation

A mechanochemical model of vascular network formation based on in vitro experiments. The model is considering endothelial cells within an extracellular matrix. Here only considered in the case of two space dimensions. Denote by \( n(x, y, t) \) the endothelial cell density, by \( \rho(x, y, t) \) the thickness of the extracellular matrix, by \( u(x, y, t) \) displacement of the matrix. The matrix is modelled as a linear viscoelastic material. Assume the cells move by convection and with an anisotropic strain-dependent random motion. The equation for the nondimensional version is written as (see e.g. [8], chapter 8)

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{u}_t) = \nabla \cdot (D(\varepsilon)n), \\
\nabla \cdot \left[ \mu_1 \varepsilon_t + \mu_2 \theta \mathbf{I} + \varepsilon + \nu \theta \mathbf{I} + \frac{\tau n \mathbf{I}}{1 + \alpha n^2} \right] = \frac{s}{\rho} \mathbf{u}_t, \\
\rho(x, y, t) = 1 - \nu \theta,
\]

(2.53)

where \( \mu_1, \mu_2, \nu, \tau, \alpha, \sigma \) are constants, \( \theta \) the dilation \( \theta = \nabla \cdot \mathbf{u} \) and \( \varepsilon \) the matrix strain tensor \( \varepsilon = (1/2)(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \). The diffusion tensor \( D(\varepsilon) \) is given by (see [8], chapter 8)

\[
D(\varepsilon) = D_0 \begin{pmatrix}
1 + (\varepsilon_{11} - \varepsilon_{22})/2 & \varepsilon_{12}/2 \\
\varepsilon_{21}/2 & 1 + (\varepsilon_{22} - \varepsilon_{11})/2
\end{pmatrix}
\]

(2.54)

The boundary conditions for a domain \( B \), with zero displacement at the boundary and zero flux at boundary for the cells. The conditions are written as (see [8], chapter 8)

\[
\mathbf{u}(x, y, t) = 0, \quad (x, y) \in \partial B, \\
n \mathbf{u}_t - \nabla \cdot (D(\varepsilon)n) = 0, \quad \mathbf{x} = (x, y) \in \partial B.
\]
2.8 Epidermal wound healing

A mechanochemical model for epidermal wound healing. The epidermis is the outer layer of the skin. Epidermal wounds close due to migration of epidermal cells. The epithelial sheet is at the front of the wound closure. Assumes that the mitotic activity near the wound is controlled by a chemical. Denote by \( n(x, t) \) the epithelial cell density and \( c(x, t) \) the concentration of the mitosis-regulating chemical. Assume the chemical disperse by diffusion and that the cell migration is model with a diffusion term. The equations in nondimensional form are written (see e.g. [8], chapter 9)

\[
\frac{\partial n}{\partial t} = D r^2 n + s(c)n(2-n) - n, \\
\frac{\partial c}{\partial t} = D c r^2 c + g(n) - \lambda c,
\]

where \( \lambda \) is a positive constant, \( g(n) \) represents production of the chemical \( c \) by the cells and \( s(c) \) represents the chemical control of mitosis. The model can be considered in the two cases: one in which the chemical \( c \) activates mitosis and one in which it inhibits mitosis. This is controlled by the form of \( s(c) \) and \( g(n) \). Suggested forms in the activator case is given by (see [8], chapter 9)

\[
s(c) = \frac{2c_m(h-\beta)c}{c_m^2+c^2} + \beta, \quad \text{where} \quad \beta = 1 + \frac{c_m^2 - 2hc_m}{(1-c_m)^2}, \\
g(n) = \frac{n(1+\alpha^2)}{n^2+\alpha^2}.
\]

In the inhibitor case the suggested forms are (see [8], chapter 9)

\[
s(c) = n, \quad g(n) = \frac{(h-1)c + h}{2(h-1)c + 1}
\]

Assuming that there is no epidermis within the wound domain at the initial stage, then the initial conditions is given by

\[
n(x, 0) = 0, \quad c(x, 0) = 0
\]

and boundary conditions are written as

\[
n(x, t) = 1, \quad c(x, t) = 1, \quad \text{for all} \ t
\]

where \( x \) is on the wound boundary.

2.8.1 Epidermal wounds in embryo

For epidermal wounds in embryo consider the embryonic epidermis as two cell layers thick. One of the layers is a cuboidal basal layer, the action cable develops in this basal layer. Here is a mechanochemical model for how the actin cable forms. Denote by
\( \mathbf{u}(\mathbf{r}, t) \) the displacement vector of the material, a point which was initially at position \( \mathbf{r} \) is after time \( t_0 \) at \( \mathbf{r} + \mathbf{u}(\mathbf{r}, t_0) \). The nondimensional version of the equations are written as (see e.g. [8], chapter 9)

\[
\nabla \cdot \left[ \frac{1}{1 + \nabla \cdot \mathbf{u}} \left( E \varepsilon + \Gamma \nabla \cdot \mathbf{u} \mathbf{I} + \frac{\mathbf{I}}{1 + \beta \nabla \cdot \mathbf{u}} \right) \right] - \frac{\lambda \mathbf{u}}{1 + \nabla \cdot \mathbf{u}} = 0 \tag{2.56}
\]

where \( E, \Gamma, \beta, \lambda \) are constants, with the restriction \( 0 < \beta < 1 \). \( \mathbf{I} \) is the identity matrix, \( \varepsilon \) is the matrix strain tensor, and is given by \( \varepsilon = (1/2)(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \).

### 2.9 Dermal wound healing

A mechanochemical model for dermal wound healing. Denote by \( n(\mathbf{r}, t) \) the cell density, \( \rho(\mathbf{r}, t) \) the density of the extracellular matrix and \( \mathbf{u}(\mathbf{x}, t) \) the ECM displacement vector. The equations are written as (see e.g. [8], chapter 9)

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{u}) = \nabla \cdot (D \nabla n) + r n (1 - n),
\]

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,
\]

\[
\nabla \cdot \left[ \mu_1 \mathbf{e}_t + \mu_2 \theta \mathbf{I} + (\mathbf{e} + \nu' \theta \mathbf{I}) + \frac{\tau \rho n}{1 + \lambda n^2} \mathbf{I} \right] = s \rho \mathbf{u},
\]

where \( r, \mu_1, \mu_2, \tau, \lambda, D, \nu' \) are constants and \( \mathbf{I} \) is the identity matrix. \( \theta = \nabla \cdot \mathbf{u} \) and \( \mathbf{e} \) is the matrix strain tensor \( \varepsilon = (1/2)(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \).

### 2.10 Tumour models

#### 2.10.1 Brain tumour model

A model of the effect of chemotherapy on the growth of brain tumour. Take the tumour to consist of two different cell types. The density for the cell types is denoted by \( c_1(\mathbf{x}, t) \) and \( c_2(\mathbf{x}, t) \). For the model there is assume to be two types of treatment. The cell type \( c_1 \) is sensible to both and \( c_2 \) is resistant to the first treatment but sensible to the second treatment. Both cell types are assumed to move by diffusion with the diffusion coefficient \( D \). The equations are written as (see e.g. [8], chapter 11)

\[
\frac{\partial c_1}{\partial t} = D \nabla^2 c_1 + r_1 c_1 - K_1(t) c_1 - K_2(t) c_1,
\]

\[
\frac{\partial c_2}{\partial t} = D \nabla^2 c_2 + r_2 c_2 - K_2(t) c_2,
\]

where the constants \( r_1, r_2 \) are the growth rates for \( c_1 \) respectively \( c_2 \). \( K_1(t) \) is the death rate per cell due to the first treatment and \( K_2(t) \) is the death rate per cell due to the
second treatment. Assuming there is \( n \) number of the first treatment and that it take place in the time intervals \( t \in [a_i, b_i], \; i = 1, \ldots, n \). And also \( m \) number of the second treatment that take place in the time intervals \( t \in [c_i, d_i], \; i = 1, \ldots, m \). Then \( K_1(t) \) and \( K_2(t) \) can be taken as step functions written as (see [8], chapter 11)

\[
K_1(t) = \begin{cases} 
   k_1 & \text{for } t \in [a_i, b_i], \\
   0 & \text{otherwise}
\end{cases}, \quad 
K_2(t) = \begin{cases} 
   k_2 & \text{for } t \in [c_i, d_i], \\
   0 & \text{otherwise}
\end{cases}
\]

where \( k_1, k_2 \) are constants. The condition of zero flux through the boundary gives the following conditions on the boundary

\[
\mathbf{n} \cdot \nabla c_1 = 0, \quad \mathbf{n} \cdot \nabla c_2 = 0
\]

where \( \mathbf{n} \) is a unit normal to the boundary.

### 2.10.2 Brain tumour model 2

A model for brain tumour. Here with two cell populations, it is used to model the heterogeneous within the tumour. Denote the cell density for the two types with \( u(x, t) \) and \( v(x, t) \). Assume that the cell type \( u \) has high growth rate and low diffusion and the cell type \( v \) has a moderate growth and high diffusion. The \( u \) cells can also mutate to form \( v \) cells. The equations for the nondimensional version are (see e.g [8], chapter 11)

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \nabla^2 u + u - \alpha u, \\
\frac{\partial v}{\partial t} &= \nu \nabla^2 v + \beta v + \alpha u,
\end{align*}
\]

(2.59)

where \( \alpha, \beta, \nu \) are constants.

### 2.10.3 Avascular tumour growth

A model for avascular tumour growth. The equations are written in nondimensional form as (see [11])

\[
\frac{\partial n}{\partial t} + v \frac{\partial n}{\partial r} = a(c)n - b(c)n^2
\]

\[
\nu \left( \frac{\partial c}{\partial t} + v \frac{\partial c}{\partial r} + b(c)c n \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \frac{\partial c}{\partial r} \right] - k(c)n
\]

(2.60)

\[
\frac{1}{r^2} \frac{\partial (r^2 v)}{\partial r} = b(c)n.
\]
where $\nu$ is a constant. Suggested forms for the functions $a(c), b(c), k(c)$ are given by (see [11])

\[
    a(c) = k_m(c) - k_d(c), \\
    b(c) = k_m(c) - (1 - \delta)k_d(c), \\
    k(c) = \beta k_m(c) + \gamma(c),
\]

where $\delta \in [0, 1]$ and $\beta$ is a constant. $k_m(c), k_d(c)$ are given by

\[
    k_m(c) = \frac{c^{m_1}}{c^{m_1} + c^{m_2}}, \quad k_d(c) = \frac{B}{A} \left( 1 - \sigma \frac{c^{m_2}}{c^{m_1} + c^{m_2}} \right),
\]

where $m_1, m_2, c, c_d, A, B > 0$ are constants and $\sigma \in [0, 1]$.

### 2.11 Epidemic models

#### 2.11.1 SI model

A model for the spatial spread of an epidemic. The model is similar to the $SIR$ model (1.40) but now the population also depend on the space variable $x$. Here the population is divided into susceptibles $S$ and infectives $I$, the population density is denoted by $S(x, t)$ and $I(x, t)$. The assumptions for the model are that susceptibles catch the disease at an average per capita rate $rI$. Infectives die at a per capita rate $a$. Both $S$ and $I$ are assumed to disperse by diffusion and with the same diffusion coefficient $D$. The equations are written as (see e.g. [8], chapter 13)

\[
    \frac{\partial S}{\partial t} = -rIS + D\nabla^2 S, \\
    \frac{\partial I}{\partial t} = rIS - aI + D\nabla^2 I,
\]

where $a, r, D$ are positive constants.

#### 2.11.2 Spread of rabies 1

A model for spatial spread of rabies. Since foxes are considered as the main cause of spatial spread of rabies, so therefore we only need to consider foxes. The population density for susceptibles and infectious foxes are denote by $S(x, t)$ and $I(x, t)$. This is a similar to the model (2.61) but with the difference that susceptible foxes is assumed to stay within their territory so there is no diffusion for the susceptibles. The one dimensional version of the equations are written as (see e.g. [8], chapter 13)

\[
    \frac{\partial S}{\partial t} = -rIS, \\
    \frac{\partial I}{\partial t} = rIS - aI + D\frac{\partial^2 I}{\partial x^2},
\]
where \( a, r, D \) are positive constants. It is also possible to include reproduction of the foxes into the model. Then taking a logistic growth, the equation for \( S \) becomes
\[
\frac{\partial S}{\partial t} = -r IS + BS \left( 1 - \frac{S}{S_0} \right),
\]
where \( B, S_0 \) are constants.

### 2.11.3 Spread of rabies 2

A model for spatial spread of rabies among foxes, an extension from the previous models. This is a type of SIR model, see section 1.5.1. The fox population divided is into the classes susceptibles \( S \), infected but noninfectious \( I \) and infectious \( R \). The class \( I \) is included since their is a long incubation period. Non rabied foxes are assumed to stay within their territory, so no diffusion for \( S \) and \( I \), but \( R \) is assumed to disperse by diffusion. Assume that there is a natural per capita death rate \( b \) for all classes and an increased per capita death rate \( \alpha \) for the infectious foxes. For susceptibles there is a per capita rate \( \beta R \) to become infected. For infected foxes there is a per capita rate \( \sigma \) to become infectious. The equations are written as (see e.g. [8], chapter 13)
\[
\begin{align*}
\frac{\partial S}{\partial T} &= aS - bS - \frac{(a - b)NS}{K} - \beta RS, \\
\frac{\partial I}{\partial T} &= -bI - \frac{(a - b)NI}{K} + \beta RS - \sigma I, \\
\frac{\partial R}{\partial T} &= -bR - \frac{(a - b)NR}{K} + \sigma I - \alpha R + D \frac{\partial ^2 R}{\partial X^2}, \\
\end{align*}
\]
where \( a, b, K, D, \beta, \sigma, \alpha \) are constants. \( N \) is the total population, given by \( N = S + I + R \).

### 2.11.4 Spread of rabies 3

Another extension to the model for rabies among foxes. Here also immunity is considered. The population is divided into the four classes susceptibles, infected but noninfectious, infectious and immune. The population density for these classes are denoted by \( S, I, R \) and \( Z \). The equations are written as (see e.g. [8], chapter 13)
\[
\begin{align*}
\frac{\partial S}{\partial T} &= (a - b) \left[ 1 - \frac{N}{K} \right] + a^* Z - \beta RS, \\
\frac{\partial I}{\partial T} &= \beta RS - \sigma I - \left[ b + (a - b) \frac{N}{K} \right] I, \\
\frac{\partial R}{\partial T} &= \sigma I - \alpha R - \gamma R - \left[ b + (a - b) \frac{N}{K} \right] R + D_R \frac{\partial ^2 R}{\partial X^2}, \\
\frac{\partial Z}{\partial T} &= \gamma R + (a - a^*) Z - \left[ b + (a - b) \frac{N}{K} \right] Z, \\
\end{align*}
\]
where \( a, a^*, b, K, \beta, \sigma, \alpha, \gamma \) are constants. \( N \) is the total population, given by \( N = S + I + R + Z \).

### 2.12 Various other models

Model for Belousov-Zhabotinskii reaction in one space dimension. Assume chemicals spread by diffusion. The equations in nondimensional form is written as (see e.g. [8], chapter 1)

\[
\frac{\partial u}{\partial t} = Lrv + u(1 - u - rv) + \frac{\partial^2 u}{\partial s^2}, \quad \frac{\partial v}{\partial t} = -Mv - buv + \frac{\partial^2 v}{\partial s^2}.
\]

where \( r, b, L, M \) are constants. The equations (2.66) is investigated for travelling wave-front solutions in [8].

Model for propagation of nerve action potentials. This is the FHN system taken without any applied current but with spatial diffusion in the transmembrane potential (see also the equations (1.35)). This is an example of a model in an excitable medium. The equations are written as (see e.g. [8], chapter 1)

\[
\frac{\partial u}{\partial t} = f(u) - v + D \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial v}{\partial t} = bu - \gamma v,
\]

where \( a, b, D, \gamma \) are constants and \( f(u) = u(a - u)(u - 1) \).

Generic excitable reaction diffusion system Winfree (1994b), investigated for spiral waves in [8]. The equations are written as (see e.g. [8], chapter 1)

\[
\frac{\partial u}{\partial t} = \nabla^2 u + \frac{1}{\varepsilon} \left( u - \frac{u^3}{3} - v \right), \quad \frac{\partial v}{\partial t} = \nabla^2 v + \varepsilon \left( u + \beta - \frac{v}{2} \right).
\]

where \( \beta, \varepsilon \) are constants.

### 3 Models formulated in terms of delay equations

#### 3.1 Biological oscillators

Delay model for the level of testosterone concentration, for more details see section 1.3.6. The equations here differs from the system (1.36) only in the delay term \( g_2 L(t - \frac{1}{\alpha}) \).
\( \tau \). The system is written as (see e.g. [7], chapter 7)

\[
\begin{align*}
\frac{dR}{dt} &= f(T) - b_1R, \\
\frac{dL}{dt} &= g_1R - b_2L, \\
\frac{dT}{dt} &= g_2L(t - \tau) - b_3T.
\end{align*}
\] (3.1)

### 3.2 Various other models

A model for population growth, with \( N(t) \) as population size the equation is written as (see e.g. [7], chapter 1)

\[
\frac{dN(t)}{dt} = N(t)[1 - N(t - T)]
\] (3.2)

where \( T \) is a positive constant.

A model for breathing. \( x(t) \) is the concentration of carbon dioxide in the blood. This concentration level is taken to control the breathing levels. The delay equation for \( x(t) \) is written in dimensionless form as (see e.g. [7], chapter 1)

\[
x'(t) = 1 - \alpha x(t) \frac{x^m(t - T)}{1 + x^m(t - T)}
\] (3.3)

where \( m \) is a positive constant and \( \alpha \) is a constant.

A model for the regulation of haematopoiesis, that is of the formation of blood cell elements in the blood. With \( c(t) \) as the concentration of cells in the blood. The delay equation is given by (see e.g. [7], chapter 1)

\[
\frac{dc}{dt} = \frac{\lambda a^m c(t - T)}{a^m + c^m(t - T)} - gc
\] (3.4)

where \( \lambda, a, m, g, T \) are positive constants. For details see Mackey and Glass (1977).

A delay model for HIV infection with drug therapy. Similar to the equations (1.43), see section 1.5.4 for more details. The equation is written as (see e.g. [7], chapter 10)

\[
\begin{align*}
\frac{dT^*}{dt} &= kT_0V_I(t - \tau) - \delta T^*, \\
\frac{dV_I}{dt} &= (1 - n_p)N\delta T^* - cV_I, \\
\frac{dV_NI}{dt} &= n_pN\delta T^* - cV_NI
\end{align*}
\] (3.5)

where \( c, k, n_p, N, T_0, \delta, \tau \) are constants.
II Symmetries of selected models

4 Symmetries

4.1 Introduction

Introduction of some of the terminology and theory by use of an example.

The admitted group for the equations

\[ u_t = f(u, v) + u_{xx}, \quad v_t = g(u, v) + d v_{xx}, \]  

(4.1)

can be found using the so called determining equations, see [1]. The admitted group is a local Lie group of continuous point transformations, here denoted by \( G \). The group \( G \) will take the equations (4.1) into the equations

\[ \bar{u}_t = f(\bar{u}, \bar{v}) + \bar{u}_{\bar{x}\bar{x}}, \quad \bar{v}_t = g(\bar{u}, \bar{v}) + d \bar{v}_{\bar{x}\bar{x}}. \]  

(4.2)

The group \( G \) generates a Lie algebra \( L_G \), see [1]. Here the Lie algebra is a vector space of linear differential operators of the form

\[ X = \xi^1(x) \frac{\partial}{\partial x^1} + \cdots + \xi^n(x) \frac{\partial}{\partial x^n}. \]

This vector space is closed under the commutator. The commutator of two operators

\[ X_a = \xi^i_a(x) \frac{\partial}{\partial x^i} \quad \text{and} \quad X_b = \xi^i_b(x) \frac{\partial}{\partial x^i}, \]

(summation in \( i \)) is defined as (see [1])

\[ [X_a, X_b] \equiv X_a X_b - X_b X_a = \left( X_a (\xi^i_a) - X_b (\xi^i_a) \right) \frac{\partial}{\partial x^i}. \]

An equivalence transformation (see [3]-[5]) takes the equations (4.1) into the equations

\[ \bar{u}_t = \bar{f}(\bar{u}, \bar{v}) + \bar{u}_{\bar{x}\bar{x}}, \quad \bar{v}_t = \bar{g}(\bar{u}, \bar{v}) + d \bar{v}_{\bar{x}\bar{x}}. \]  

(4.3)

The difference here to the equations (4.2) is that the functions \( f \) and \( g \) now could have different forms. A continuous equivalence group \( E_c \) for the equations (4.1) can be
found using the infinitesimal methods [3]-[5]. The Lie algebra generated by \( E_c \) is here denoted by \( L_{E} \).

From the Lie algebra \( L_{E} \) it is possible to find the Lie algebra that generates the admitted group for the equations (4.1) (see [4]), this Lie algebra is called the principal Lie algebra and denoted by \( L_{P} \). \( L_{P} \) is a subgroup of \( L_{E} \). An example of the calculations is given in section 4.8.1.

### 4.2 Equivalence group for a predator-prey model

Calculation of the equivalence group \( E_c \) for the predator prey model with equations (2.14). The equations are written here as

\[
\begin{align*}
    u_t - (c_1 + h_1 v_x)u_x - h_1 v_{xx}u - f(u, v) &= 0, \\
    v_t - (c_2 - h_2 u_x)v_x + h_2 u_{xx}v - g(u, v) &= 0.
\end{align*}
\]  

(4.4a)

Here with the two arbitrary functions \( f(u, v) \) and \( g(u, v) \). Also need for the calculation of the equivalence group is the equations

\[
\begin{align*}
    f_t &= f_x = 0, \\
    g_t &= g_x = 0.
\end{align*}
\]  

(4.4b)

Following the theory [1],[3]-[5], to find the equivalence group \( E_c \), we search for the equivalence operator \( Y \) in the form

\[
Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v} + \mu^1 \frac{\partial}{\partial f} + \mu^2 \frac{\partial}{\partial g}
\]  

(4.5)

where the functions \( \xi^i \) and \( \eta^j \) depend on the variables \((t, x, u, v)\). Also \( \mu^k \) are functions of the variables \((t, x, u, v, f, g)\). The operator \( Y \) after the necessary prolongation should satisfy the following invariance test (see [1]) on the equations (4.4)

\[
\begin{align*}
    \tilde{Y} [u_t - (c_1 + h_1 v_x)u_x - h_1 v_{xx}u - f] \big|_{(4.4)} &= 0, \\
    \tilde{Y} [v_t - (c_2 - h_2 u_x)v_x + h_2 u_{xx}v - g] \big|_{(4.4)} &= 0, \\
    \tilde{Y} [f_t] \big|_{(4.4)} &= 0, \\
    \tilde{Y} [f_x] \big|_{(4.4)} &= 0, \\
    \tilde{Y} [g_t] \big|_{(4.4)} &= 0, \\
    \tilde{Y} [g_x] \big|_{(4.4)} &= 0.
\end{align*}
\]  

(4.6)

\( \big|_{(4.4)} \) means evaluated on the system (4.4). \( \tilde{Y} \) is the prolonged operator, here \( \tilde{Y} \) is written as

\[
\tilde{Y} = Y + \zeta^1 \frac{\partial}{\partial u_t} + \zeta^2 \frac{\partial}{\partial u_x} + \zeta^3 \frac{\partial}{\partial v_t} + \zeta^4 \frac{\partial}{\partial v_x} + \zeta^5 \frac{\partial}{\partial u_{xx}} + \zeta^6 \frac{\partial}{\partial v_{xx}} + \\
+ \omega^1 \frac{\partial}{\partial f_t} + \omega^2 \frac{\partial}{\partial f_x} + \omega^3 \frac{\partial}{\partial g_t} + \omega^4 \frac{\partial}{\partial g_x}
\]  

(4.7)
where the $\zeta^i_j$ and $\omega^i_j$ are prolongation terms. Using the following notations for the variables and derivatives

\[
(t, x) = (x^1, x^2), \quad (u, v) = (u^1, u^2), \quad (f, g) = (f^1, f^2),
\]

\[
(u^i_1, u^i_2) = (u^i_1, u^i_2), \quad (f^i_1, f^i_2, f^i_3, f^i_4) = (f^i_1, f^i_2, f^i_3, f^i_4),
\]

for $i, j = 1, 2$. Then the prolongation terms (see [1]) in (4.7) are written as

\[
\zeta^i_j = D_j(\eta^i) - u^i_1 D_j(\xi^1) - u^i_2 D_j(\xi^2),
\]

\[
\zeta^i_{22} = D_2(\zeta^i_j) - u^i_{21} D_2(\xi^1) - u^i_{22} D_2(\xi^2),
\]

\[
\omega^i_k = \tilde{D}_k(\mu^i) - f^i_1 \tilde{D}_k(\xi^1) - f^i_2 \tilde{D}_k(\xi^2) - f^i_3 \tilde{D}_k(\eta^1) - f^i_4 \tilde{D}_k(\eta^2).
\]

The operators $D_1, D_2$ denote the total derivatives (see [1]) with respect to $t$ and $x$. $D_1, D_2$ are given by

\[
D_1 = \frac{\partial}{\partial x^1} + u^1_1 \frac{\partial}{\partial u^1} + u^2_1 \frac{\partial}{\partial u^2} + u^1_2 \frac{\partial}{\partial u^1_2} + \cdots
\]

The differential operators $\tilde{D}_1, \ldots, \tilde{D}_4$ are given by

\[
\tilde{D}_1 = \frac{\partial}{\partial x^1} + f^1_1 \frac{\partial}{\partial f^1} = \frac{\partial}{\partial x^1}, \quad \tilde{D}_2 = \frac{\partial}{\partial x^2} + f^2_2 \frac{\partial}{\partial f^2} = \frac{\partial}{\partial x^2},
\]

\[
\tilde{D}_3 = \frac{\partial}{\partial u^1} + f^3_1 \frac{\partial}{\partial f^1} = \frac{\partial}{\partial u^1} + f^3_1 \frac{\partial}{\partial f^1} + f^3_2 \frac{\partial}{\partial f^2},
\]

\[
\tilde{D}_4 = \frac{\partial}{\partial u^2} + f^4_3 \frac{\partial}{\partial f^3} = \frac{\partial}{\partial u^2} + f^4_1 \frac{\partial}{\partial f^1} + f^4_2 \frac{\partial}{\partial f^2}.
\]

From the equations (4.6) it can be found after some calculations that, the Lie algebra $L_{\xi}$ is spanned by the three operators

\[
Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}.
\]

### 4.3 Equivalence group for the FitzHugh-Nagumo model

Calculation of the equivalence group for the equations (2.67). The equations are here written as

\[
u_t - f(u) + v - D u_{xx} = 0, \quad v_t - bu + \gamma v = 0
\]

where $f(u)$ is an arbitrary function. Also need for the calculation of the equivalence group is the equations

\[
f_t = f_x = f_v = 0.
\]
Following the theory \cite{[1],[3]-[5]}, to find the equivalence group $E_c$, we search for its generator $Y$ in the form

$$
Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v} + \mu \frac{\partial}{\partial f} \tag{4.10}
$$

where the functions $\xi^i$ and $\eta^j$ depend on the variables $(t, x, u, v)$. Also $\mu$ is a function of the variables $(t, x, u, v, f)$. The operator $Y$ after the necessary prolongation should satisfy the following invariance test (see \cite{[1]}) on the equations (4.9)

$$
\tilde{Y} [u_t - f(u) + v - Du_{xx}] |_{(4.9)} = 0,
\tilde{Y} [v_t - bu + \gamma v] |_{(4.9)} = 0,
\tilde{Y} [f_u] |_{(4.9)} = 0,
\tilde{Y} [f_v] |_{(4.9)} = 0,
\tilde{Y} [f_f] |_{(4.9)} = 0, \tag{4.11}
$$

where $\tilde{Y}$ is the prolonged operator. Here $\tilde{Y}$ is written as

$$
\tilde{Y} = Y + \zeta^1 \frac{\partial}{\partial u_t} + \zeta^2 \frac{\partial}{\partial v_t} + \zeta^3 \frac{\partial}{\partial u_{xx}} + \omega_1 \frac{\partial}{\partial f_t} + \omega_2 \frac{\partial}{\partial f_x} + \omega_3 \frac{\partial}{\partial f_v}. \tag{4.12}
$$

The prolongation terms (see \cite{[1]}) in (4.12) are written as

$$
\begin{align*}
\zeta^1 &= D_t(\eta^1) - u_tD_t(\xi^1) - u_xD_x(\xi^2), \\
\zeta^2 &= D_t(\eta^2) - v_tD_t(\xi^1) - v_xD_x(\xi^2), \\
\zeta^3 &= D_x(\zeta^3) - u_xD_x(\xi^1) - u_{xx}D_{xx}(\xi^2), \\
\omega_1 &= D_t(\mu) - f_tD_t(\xi^1) - f_xD_x(\xi^1) - f_uD_u(\eta^1) - f_vD_v(\eta^2), \\
\omega_2 &= D_x(\mu) - f_tD_t(\xi^1) - f_xD_x(\xi^2) - f_uD_u(\eta^1) - f_vD_v(\eta^2), \\
\omega_3 &= D_v(\mu) - f_tD_v(\xi^1) - f_xD_v(\xi^2) - f_uD_u(\eta^1) - f_vD_v(\eta^2).
\end{align*}
$$

The operators $D_t$, $D_x$ denote the total derivatives (see \cite{[1]}) and are given by

$$
\begin{align*}
D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + v_t \frac{\partial}{\partial v} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots, \\
D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + \cdots.
\end{align*}
$$

The differential operators $\tilde{D}_t$, $\tilde{D}_x$, $\tilde{D}_u$, $\tilde{D}_v$ are given by

$$
\begin{align*}
\tilde{D}_t &= \frac{\partial}{\partial t} + f_t \frac{\partial}{\partial f} = \frac{\partial}{\partial t}, \\
\tilde{D}_x &= \frac{\partial}{\partial x} + f_x \frac{\partial}{\partial f} = \frac{\partial}{\partial x}, \\
\tilde{D}_u &= \frac{\partial}{\partial u} + f_u \frac{\partial}{\partial f} = \frac{\partial}{\partial u}, \\
\tilde{D}_v &= \frac{\partial}{\partial v} + f_v \frac{\partial}{\partial f} = \frac{\partial}{\partial v}.
\end{align*}
$$
From the equations (4.11) it can be found after some calculations that, the Lie algebra $L_E$ is spanned by the four operators

$$Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} + f \frac{\partial}{\partial f}, \quad Y_4 = \frac{\partial}{\partial u} + b \frac{\partial}{\partial v} + b \frac{\partial}{\partial f}. \quad (4.13)$$

Projection on operators (see [3],[4]) are denoted according to the following rules

$$X_i = \text{pr}_{(t,x,u,v)} Y_i, \quad Z_i = \text{pr}_{(u,f)} Y_i.$$  

As mentioned in section 4.1 it is now possible to find the principal algebra. Here $L_P$ is found to be spanned by

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}.$$  

### 4.3.1 Extension of principal Lie algebra with $Y_5$

Taking an operator $Y_5$ so that $Y_1, Y_2$ and $Y_5$ span a Lie algebra (subalgebra of $L_E$). Here taking $Y_5$ in the form $Y_5 = Y_1 + \gamma Y_3 + \lambda_4 Y_4$

$$Y_5 = \frac{\partial}{\partial t} + (\gamma u + \lambda_4) \frac{\partial}{\partial u} + (\gamma v + \frac{\lambda_4 b}{\gamma}) \frac{\partial}{\partial v} + (\gamma f + \frac{\lambda_4 b}{\gamma}) \frac{\partial}{\partial f}$$

where $\gamma$ and $\lambda_4$ are constants. To find the equations from (4.9) that admits the operator

$$X_5 = \frac{\partial}{\partial t} + (\gamma u + \lambda_4) \frac{\partial}{\partial u} + (\gamma v + \frac{\lambda_4 b}{\gamma}) \frac{\partial}{\partial v} \quad (4.14)$$

the theory from [3],[4] is used. This lead us to investigate the conditions for the following equation

$$f = F(u) \quad (4.15)$$

to be invariant with respect to $Z_5 = \text{pr}_{(u,f)} Y_5$. Here it is found that

$$f = (\gamma u + \lambda_4)(K_1 + 1) - \frac{\lambda_4 b}{\gamma^2} \quad (4.16)$$

where $K_1$ is an arbitrary constants. Then the equations (4.9) after the substitutions (4.16) are written as

$$u_t - (\gamma u + \lambda_4)(K_1 + 1) + \frac{\lambda_4 b}{\gamma^2} + v - Du_{xx} = 0, \quad v_t + bu + \gamma v = 0. \quad (4.17)$$

So from the equivalence algebra $L_E$, this are the equations that admits the operators $X_1, X_2, X_5$.  

4.4 Equivalence group for a general cell-chemotaxis model

Using the theory of Lie group analysis to calculating the continuous equivalence group for the equations (2.34) in one space dimension. To be able to denote derivatives with indices lets set \( D_n(c) = d(c) \). The equations are then written as

\[
\begin{align*}
n_t - d(c)n_x &- d n_{xx} + n_x f c_x + n c_x (f n_x + f c_x) + \\
+ n f c_{xx} - h &= 0 \\
c_t - D c c_{xx} - g &= 0.
\end{align*}
\] (4.18a)

Here with the four arbitrary functions \( f(n, c), h(n, c), g(n, c) \) and \( d(c) \). Also needed for the calculations is the following equations

\[
\begin{align*}
f_t &= f_x = 0, \\
h_t &= h_x = 0, \\
g_t &= g_x = 0, \\
d_t &= d_x = d_n = 0.
\end{align*}
\] (4.18b)

According to the theory \([1],[3]-[5]\) we should try to find generators \( Y \) of the continuous equivalence group in the form

\[
Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial n} + \eta^2 \frac{\partial}{\partial c} + \mu^1 \frac{\partial}{\partial f} + \mu^2 \frac{\partial}{\partial h} + \mu^3 \frac{\partial}{\partial g} + \mu^4 \frac{\partial}{\partial d}
\] (4.19)

where \( \xi^i \) and \( \eta^j \) are functions of the variables \((t, x, n, c)\) and \( \mu^k \) are functions of the variables \((t, x, n, c, f, h, g, d)\). The invariance test (see \([1]\)) of the equations (4.18) is written as

\[
\begin{align*}
\overline{Y} \left[ n_t - d c x n_x - d n_{xx} + n_x f c_x + n c_x (f n_x + f c_x) + \\
+ n f c_{xx} - h \right] \big|_{(4.18)} &= 0, \\
\overline{Y} \left[ c_t - D c c_{xx} - g \right] \big|_{(4.18)} &= 0, \\
\overline{Y} \left[ f_t \right] \big|_{(4.18)} &= 0, \\
\overline{Y} \left[ f_x \right] \big|_{(4.18)} &= 0, \\
\overline{Y} \left[ h_t \right] \big|_{(4.18)} &= 0, \\
\overline{Y} \left[ h_x \right] \big|_{(4.18)} &= 0, \\
\overline{Y} \left[ g_t \right] \big|_{(4.18)} &= 0, \\
\overline{Y} \left[ g_x \right] \big|_{(4.18)} &= 0, \\
\overline{Y} \left[ d_t \right] \big|_{(4.18)} &= 0, \\
\overline{Y} \left[ d_x \right] \big|_{(4.18)} &= 0, \\
\overline{Y} \left[ d_n \right] \big|_{(4.18)} &= 0,
\end{align*}
\] (4.20)

where \( \overline{Y} \) is the prolonged operator. Here \( \overline{Y} \) is written as

\[
\overline{Y} = Y + \zeta_1^1 \frac{\partial}{\partial n_t} + \zeta_1^2 \frac{\partial}{\partial c_t} + \zeta_1^3 \frac{\partial}{\partial n_x} + \zeta_2^1 \frac{\partial}{\partial n_{xx}} + \zeta_2^2 \frac{\partial}{\partial c_{xx}} + \zeta_2^3 \frac{\partial}{\partial n_{x}} + \zeta_2^4 \frac{\partial}{\partial c_{x}} + \\
+ \omega_1^1 \frac{\partial}{\partial f_t} + \omega_1^2 \frac{\partial}{\partial f_x} + \omega_1^3 \frac{\partial}{\partial f_n} + \omega_1^4 \frac{\partial}{\partial f_c} + \omega_2^1 \frac{\partial}{\partial h_t} + \omega_2^2 \frac{\partial}{\partial h_x} + \omega_2^3 \frac{\partial}{\partial h_n} + \omega_2^4 \frac{\partial}{\partial h_c} + \\
+ \omega_3^1 \frac{\partial}{\partial g_t} + \omega_3^2 \frac{\partial}{\partial g_x} + \omega_3^3 \frac{\partial}{\partial g_n} + \omega_3^4 \frac{\partial}{\partial g_c}.
\] (4.21)
Using the following notations for the variables and derivatives

\[(t, x) = (x^1, x^2), \quad (n, c) = (u^1, u^2), \quad (f, h, g, d) = (f^1, f^2, f^3, f^4), \]

\[(u^i_t, u^i_x) = (u^i_1, u^i_2), \quad (f^i_t, f^i_x, f^i_n, f^i_c) = (f^i_2, f^i_3, f^i_4), \]

for \(i = 1, 2\) and \(j = 1, \ldots, 4\). Then the prolongation terms (see [1]) in (4.21) are written as

\[
\begin{align*}
\zeta^j_i &= D_j(\eta^i) - u^1_i D_j(\xi^1) - u^2_i D_j(\xi^2), \\
\zeta^j_{22} &= D_2(\zeta^j_2) - u^1_{21} D_2(\xi^1) - u^2_{22} D_2(\xi^2), \\
\omega_k^j &= \tilde{D}_k(\mu^j) - f^i_{1k} \tilde{D}_k(\xi^i) - f^i_{2k} \tilde{D}_k(\xi^i) - f^i_{3k} \tilde{D}_k(\eta^i) - f^i_{4k} \tilde{D}_k(\eta^i).
\end{align*}
\]

The operators \(D_1, D_2\) denoting total derivatives (see [1]) are given by

\[
D_i = \frac{\partial}{\partial x^i} + u^1_i \frac{\partial}{\partial u^1} + u^2_i \frac{\partial}{\partial u^2} + u^1_1 \frac{\partial}{\partial u^1_1} + u^1_2 \frac{\partial}{\partial u^1_2} + \cdots.
\]

The differential operators \(\tilde{D}_1, \ldots, \tilde{D}_4\) are given by

\[
\begin{align*}
\tilde{D}_1 &= \frac{\partial}{\partial x^1} + f^1_1 \frac{\partial}{\partial f^1} = \frac{\partial}{\partial x^1}, \\
\tilde{D}_2 &= \frac{\partial}{\partial x^2} + f^2_2 \frac{\partial}{\partial f^2} = \frac{\partial}{\partial x^2}, \\
\tilde{D}_3 &= \frac{\partial}{\partial u^1} + f^3_1 \frac{\partial}{\partial f^3} = \frac{\partial}{\partial u^1} + f^3_2 \frac{\partial}{\partial f^3} + f^3_3 \frac{\partial}{\partial f^3}, \\
\tilde{D}_4 &= \frac{\partial}{\partial u^2} + f^4_1 \frac{\partial}{\partial f^4} = \frac{\partial}{\partial u^2} + f^4_1 \frac{\partial}{\partial f^4} + f^4_2 \frac{\partial}{\partial f^4} + f^4_3 \frac{\partial}{\partial f^4}.
\end{align*}
\]

From the equations (4.21) it can be found after some calculations that, the Lie algebra \(L_{\xi}\) is spanned by the eight operators

\[
\begin{align*}
Y_1 &= t \frac{\partial}{\partial t} + \frac{1}{2} x \frac{\partial}{\partial x} - h \frac{\partial}{\partial h} - g \frac{\partial}{\partial g}, \\
Y_2 &= \frac{\partial}{\partial t}, \\
Y_3 &= \frac{\partial}{\partial x}, \\
Y_4 &= n \frac{\partial}{\partial n} + h \frac{\partial}{\partial h}, \\
Y_5 &= -c \frac{\partial}{\partial n} + \frac{cf - d + D_c}{n} \frac{\partial}{\partial f} - g \frac{\partial}{\partial h}, \\
Y_6 &= \frac{\partial}{\partial n} - \frac{f}{n} \frac{\partial}{\partial f}, \\
Y_7 &= c \frac{\partial}{\partial c} - f \frac{\partial}{\partial f} + g \frac{\partial}{\partial g}, \\
Y_8 &= \frac{\partial}{\partial c}.
\end{align*}
\]

The commutator table for the equivalence algebra \(L_{\xi}\) (4.22) is given by

<table>
<thead>
<tr>
<th>([Y_1, Y_2])</th>
<th>Y_1</th>
<th>Y_2</th>
<th>Y_3</th>
<th>Y_4</th>
<th>Y_5</th>
<th>Y_6</th>
<th>Y_7</th>
<th>Y_8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y_1</td>
<td>0</td>
<td>-Y_2</td>
<td>-Y_3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Y_2</td>
<td>Y_2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Y_3</td>
<td>\frac{1}{2} Y_3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Y_4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-Y_5</td>
<td>-Y_6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Y_5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Y_5</td>
<td>0</td>
<td>0</td>
<td>-Y_5</td>
<td>Y_6</td>
</tr>
<tr>
<td>Y_6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Y_6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Y_7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Y_5</td>
<td>0</td>
<td>0</td>
<td>-Y_8</td>
</tr>
<tr>
<td>Y_8</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-Y_6</td>
<td>0</td>
<td>Y_8</td>
<td>0</td>
</tr>
</tbody>
</table>
Projection on operators (see [3],[4]) are denoted according to the following rules

\[ X_i = \text{pr}_{(t,x,n,c)} Y_i, \quad Z_i = \text{pr}_{(n,c,f,h,g,d)} Y_i. \]

The principal Lie algebra \( L_P \) is found to be spanned by the two operators

\[ X_2 = \frac{\partial}{\partial t}, \quad X_3 = \frac{\partial}{\partial x}. \] (4.23)

### 4.4.1 Extension of principal Lie algebra with \( Y_9 \)

Taking an operator \( Y_9 \) so that \( Y_2, Y_3 \) and \( Y_9 \) span a Lie algebra (subalgebra of \( L_E \)). Here taking \( Y_9 \) in the form

\[ Y_9 = \lambda_1 Y_1 + \lambda_3 Y_3 = \lambda_1 \frac{\partial}{\partial t} + \lambda_3 \frac{\partial}{\partial x} + \lambda_4 n \frac{\partial}{\partial n} + (\lambda_4 - \lambda_1) h \frac{\partial}{\partial h} - \lambda_1 g \frac{\partial}{\partial g} \]

where \( \lambda_1 \) and \( \lambda_4 \) are constants. Then the question is how to find those equations from (4.18a) that admits the operators

\[ X_2 = \frac{\partial}{\partial t}, \quad X_3 = \frac{\partial}{\partial x}, \quad X_9 = \lambda_1 \frac{\partial}{\partial t} + \lambda_3 \frac{\partial}{\partial x} + \lambda_4 n \frac{\partial}{\partial n}. \] (4.24)

This can be done using the theory from [3],[4], starting with finding the equations

\[ f = F(n, c), \quad h = H(n, c), \quad g = G(n, c), \quad d = D(c) \] (4.25)

which are invariant with respect to \( Z_9 \). In this case we find

\[ f = F_1(c), \quad g = F_2(c)n^{(-\lambda_1/\lambda_4)}, \quad h = F_3(c)n^{\frac{\lambda_4 - \lambda_1}{\lambda_4}}, \quad d = D(c) \] (4.26)

where \( F_1(c), F_2(c), F_3(c) \) and \( D(c) \) are arbitrary functions. Then the equations (4.18a) after the substitutions (4.26) are written as

\[
\begin{align*}
  n_t - D'(c)c_x n_x - D(c)n_{xx} + F_1(c)n_x e_x + F'_1(c)n c_x^2 + \\
  + F_1(c)n c_{xx} - F_3(c)n^{\frac{\lambda_4 - \lambda_1}{\lambda_4}} = 0,
\end{align*}
\]

(4.27)

\[
\begin{align*}
  e_t - D e c_{xx} - F_2(c)n^{(-\lambda_1/\lambda_4)} = 0.
\end{align*}
\]

So from the equivalence algebra \( L_E \), this are the equations that admits the operators \( X_2, X_3, X_9 \).
4.4.2 Extension of principal Lie algebra with $Y_{10}$

Taking an operator $Y_{10}$ so that $Y_2, Y_3$ and $Y_{10}$ span a Lie algebra (subalgebra of $L_\mathcal{E}$). Here taking $Y_{10}$ in the form

$$Y_{10} = \lambda_2 Y_2 + Y_4 + Y_7 = \lambda_2 \frac{\partial}{\partial t} + n \frac{\partial}{\partial n} + c \frac{\partial}{\partial c} - f \frac{\partial}{\partial f} + g \frac{\partial}{\partial g} + h \frac{\partial}{\partial h}$$

where $\lambda_2$ is a constant. Then the question is how to find those equations from (4.18a) that admits the operators

$$X_2 = \frac{\partial}{\partial t}, \quad X_3 = \frac{\partial}{\partial x}, \quad X_{10} = \lambda_2 \frac{\partial}{\partial t} + n \frac{\partial}{\partial n} + c \frac{\partial}{\partial c} - f \frac{\partial}{\partial f} + g \frac{\partial}{\partial g} + h \frac{\partial}{\partial h}. \quad (4.28)$$

This can be done using the theory from [3],[4], starting with finding the equations

$$f = F(n, c), \quad h = H(n, c), \quad g = G(n, c), \quad d = D(c) \quad (4.29)$$

which are invariant with respect to $Z_{10}$. In this case we find

$$f = \frac{1}{n} F_1 \left( \frac{c}{n} \right), \quad g = n F_2 \left( \frac{c}{n} \right), \quad h = n F_3 \left( \frac{c}{n} \right), \quad d = C_1 \quad (4.30)$$

where $F_1(c/n), F_2(c/n)$ and $F_3(c/n)$ are arbitrary functions. Then the equations (4.18a) after the substitutions (4.30) are written as

$$n_t - C_1 n_{xx} + \frac{1}{n} F_1 \left( \frac{c}{n} \right) n_x c_x - \frac{n_x c_x}{n} F_1 \left( \frac{c}{n} \right) + c_x F_1' \left( \frac{c}{n} \right) \left[ \frac{n c_x - c n_x}{n^2} \right]$$

$$+ F_1 \left( \frac{c}{n} \right) c_{xx} - n F_3 \left( \frac{c}{n} \right) = 0,$$

$$c_t - D_c c_{xx} - n F_2 \left( \frac{c}{n} \right) = 0. \quad (4.31)$$

So from the equivalence algebra $L_\mathcal{E}$, this are the equations that admits the operators $X_2, X_3$ and $X_{10}$.

4.5 Admitted group for a chemotaxis model

4.5.1 One space dimension

Calculation of the admitted group for the equations (2.35) in one-dimension. The equations are here written as

$$n_t - D_n n_{xx} + \alpha n_x c_x + \alpha n c_{xx} - r n \left( 1 - \frac{n}{n_0} \right) = 0 \quad (4.32)$$

$$c_t - D_c c_{xx} - \frac{\nu n}{n + \gamma} + \mu c = 0.$$

The admitted group for equations (4.32) is generated by the three operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = e^{-nt} \frac{\partial}{\partial c}. \quad (4.33)$$
### 4.5.2 Two space dimensions

Calculation of the admitted group for the equations (2.35) in two space dimensions. The equations are here written as

\[
\begin{align*}
    n_t &= D_n(n_{xx} + n_{yy}) + \alpha(n_x c_x + n c_{xx} + n_y c_y + n c_{yy}) - r n \left(1 - \frac{n}{n_0}\right) = 0, \\
    c_t &= D_c(c_{xx} + c_{yy}) - \frac{\nu n}{n + \gamma} + \mu c = 0.
\end{align*}
\]

(4.34)

The admitted group for the equations (4.34) is generated by the five operators

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial x}, \quad X_4 = \frac{\partial}{\partial y}, \quad X_5 = e^{-\mu t} \frac{\partial}{\partial c}.
\]

The commutator table for the Lie algebra spanned by \(X_1, \ldots, X_5\) is given by

\[
\begin{array}{c|ccccc}
[X_r, X_c] & X_1 & X_2 & X_3 & X_4 & X_5 \\
\hline
X_1 & 0 & 0 & 0 & 0 & -X_5 \\
X_2 & 0 & 0 & X_4 & -X_3 & 0 \\
X_3 & 0 & -X_4 & 0 & 0 & 0 \\
X_4 & 0 & X_3 & 0 & 0 & 0 \\
X_5 & X_5 & 0 & 0 & 0 & 0 \\
\end{array}
\]

### 4.6 Equivalence group of a reaction diffusion model

#### 4.6.1 One space dimension

Calculating the equivalence algebra for the equations (2.23) in one dimension. The equations are here written as

\[
\begin{align*}
    u_t &= \gamma f(u, v) + u_{xx}, \\
    v_t &= \gamma g(u, v) + dv_{xx}
\end{align*}
\]

(4.35a)

where \(f(u, v)\) and \(g(u, v)\) are two arbitrary functions. Also need for the calculation of the equivalence group is the equations

\[
\begin{align*}
    f_t = f_x = 0, & \quad g_t = g_x = 0.
\end{align*}
\]

(4.35b)

According to the theory [1],[3]-[5]. To find the equivalence group \(E_c\), we search for the equivalence operator \(Y\) in the form

\[
Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v} + \mu^1 \frac{\partial}{\partial f} + \mu^2 \frac{\partial}{\partial g}
\]

(4.36)
where the functions $\xi^i$ and $\eta^j$ depend on the variables $(t, x, u, v)$. Also $\mu^k$ are functions of the variables $(t, x, u, v, f, g)$. The operator $Y$ after the necessary prolongation should satisfy the following invariance test (see [1]) on the equations (4.35)

$$
\tilde{Y}[u_t - \gamma f - u_{xx}]|_{(4.35)} = 0,
$$
$$
\tilde{Y}[v_t - \gamma g - d v_{xx}]|_{(4.35)} = 0,
$$
$$
\tilde{Y}[f_x]|_{(4.35)} = 0, \quad \tilde{Y}[f_x]|_{(4.35)} = 0,
$$
$$
\tilde{Y}[g_x]|_{(4.35)} = 0, \quad \tilde{Y}[g_x]|_{(4.35)} = 0
$$

(4.37)

where $\tilde{Y}$ is the prolonged operator. Here $\tilde{Y}$ is written as

$$
\tilde{Y} = Y + \zeta^1 \frac{\partial}{\partial u} + \zeta^2 \frac{\partial}{\partial v} + \zeta^3 \frac{\partial}{\partial u_{xx}} + \omega^1 \frac{\partial}{\partial f_t} + \omega^2 \frac{\partial}{\partial f_x} + \omega^3 \frac{\partial}{\partial g_t} + \omega^4 \frac{\partial}{\partial g_x}.
$$

(4.38)

From the equations (4.37) it can be found after some calculations that, the Lie algebra $L_E$ is spanned by the seven operators

$$
Y_1 = v \frac{\partial}{\partial v} + g \frac{\partial}{\partial g}, \quad Y_2 = \frac{\partial}{\partial v}, \quad Y_3 = u \frac{\partial}{\partial u} + f \frac{\partial}{\partial f}, \quad Y_4 = \frac{\partial}{\partial u},
$$
$$
Y_5 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2f \frac{\partial}{\partial f} - 2g \frac{\partial}{\partial g}, \quad Y_6 = \frac{\partial}{\partial t}, \quad Y_7 = \frac{\partial}{\partial x}.
$$

(4.39)

If $d = 1$ in the equations (4.35a) then the Lie algebra $L_E$ (4.39) is extended with the two operators

$$
Y_{d1} = v \frac{\partial}{\partial u} + g \frac{\partial}{\partial f}, \quad Y_{d2} = u \frac{\partial}{\partial v} + f \frac{\partial}{\partial g}.
$$

Projection on operators (see [3],[4]) are done according to the following rules

$$
X_i = \text{pr}_{(t,x,u,v)} Y_i, \quad Z_i = \text{pr}_{(u,v,f,g)} Y_i.
$$

(4.40)

The principal Lie algebra $L_P$ is found to be spanned by the two operators

$$
X_6 = \frac{\partial}{\partial t}, \quad X_7 = \frac{\partial}{\partial x}.
$$

(4.41)

### 4.7 Extension of principal Lie algebra with $Y_8$

Taking an operator $Y_8$ so that $Y_6, Y_7$ and $Y_8$ span a Lie algebra (subalgebra of $L_E$). Here taking $Y_8 = \lambda_1 Y_1 + \lambda_2 Y_2 + \lambda_3 Y_3$, then $Y_8$ is written as

$$
Y_8 = 2\lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial}{\partial x} + (\lambda_1 v + \lambda_2) \frac{\partial}{\partial v} - 2\lambda_3 f \frac{\partial}{\partial f} + (\lambda_1 - 2\lambda_3) g \frac{\partial}{\partial g}.
$$
where $\lambda_1, \lambda_2$ and $\lambda_5$ are constants. Then the question is how to find those equations from (4.35a) that admits the operators (4.41) and

$$X_8 = 2\lambda_5 t \frac{\partial}{\partial t} + \lambda_5 x \frac{\partial}{\partial x} + (\lambda_1 v + \lambda_2) \frac{\partial}{\partial v}. \quad (4.42)$$

This can be done using the theory [3],[4], by first finding the equations

$$f = F(u, v), \quad g = G(u, v) \quad (4.43)$$

which are invariant with respect to $Z_8$. In this case we find

$$f = F_1(u)(\lambda_1 v + \lambda_2)^{-\frac{2\lambda_5}{\lambda_1}}, \quad g = F_2(u)(\lambda_1 v + \lambda_2)^{\frac{\lambda_1 - 2\lambda_5}{\lambda_1}} \quad (4.44)$$

where $F_1(u)$ and $F_2(u)$ are arbitrary functions. Then the equations (4.35a) after the substitutions (4.44) are written as

$$u_t = \gamma F_1(u)(\lambda_1 v + \lambda_2)^{-\frac{2\lambda_5}{\lambda_1}} + u_{xx},$$

$$v_t = \gamma F_2(u)(\lambda_1 v + \lambda_2)^{\frac{\lambda_1 - 2\lambda_5}{\lambda_1}} + dv_{xx}. \quad (4.45)$$

So from the equivalence algebra $L_\mathcal{E}$, this are the equations that admits the operators $X_6, X_7$ and $X_8$.

### 4.8 Extension of principal Lie algebra with $Y_9$

Here taking $Y_9 = Y_1 + Y_3 + KY_4 + Y_6$, then $Y_9$ is written as

$$Y_9 = \frac{\partial}{\partial t} + (u + K) \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} + f \frac{\partial}{\partial f} + g \frac{\partial}{\partial g}$$

where $K$ is a constant. Then the question is how to find those equations from (4.35a) that admits the operators (4.41) and

$$X_9 = \frac{\partial}{\partial t} + (u + K) \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}. \quad (4.46)$$

This can be done using the theory [3],[4], by first finding the equations

$$f = F(u, v), \quad g = G(u, v) \quad (4.47)$$

which are invariant with respect to $Z_9$. In this case we find

$$f = F \left( \frac{v}{u + K} \right) [u + K], \quad g = G \left( \frac{v}{u + K} \right) [u + K] \quad (4.48)$$
where \( F \) and \( G \) are arbitrary functions. Then the equations (4.35a) after the substitutions (4.48) are written as

\[
\begin{align*}
    u_t &= \gamma F \left( \frac{v}{u + K} \right) [u + K] + u_{xx}, \\
    v_t &= \gamma G \left( \frac{v}{u + K} \right) [u + K] + dv_{xx}.
\end{align*}
\]

(4.49)

So from the equivalence algebra \( \mathcal{L_E} \), this are the equations that admits the operators \( X_6, X_7 \) and \( X_9 \).

### 4.8.1 Two space dimensions

Calculation of equivalence group for the equations (2.23) in two-dimensions, here written as

\[
\begin{align*}
    u_t &= f(u,v) + u_{xx} + u_{yy}, \\
    v_t &= g(u,v) + d(v_{xx} + v_{yy})
\end{align*}
\]

(4.50a)

where \( f(u,v) \) and \( g(u,v) \) are two arbitrary functions. Also need for the calculation of the equivalence group is the equations

\[
\begin{align*}
    f_x &= f_y = f_t = 0, \\
    g_x &= g_y = g_t = 0.
\end{align*}
\]

(4.50b)

According to the theory [1],[3]-[5]. To find the equivalence group \( \mathcal{E_c} \), we search for the equivalence operator \( \mathcal{Y} \) in the form

\[
\begin{align*}
    \mathcal{Y} &= \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v} + \mu^1 \frac{\partial}{\partial f} + \mu^2 \frac{\partial}{\partial g}.
\end{align*}
\]

(4.51)

where the functions \( \xi^i \) and \( \eta^j \) are functions of the variables \((t, x, y, u, v)\). Also \( \mu^k \) are functions of variables \((t, x, y, u, v, f, g)\). The operator \( \mathcal{Y} \) after the necessary prolongation should satisfy the following invariance test (see [1]) on the equations (4.50)

\[
\begin{align*}
    \mathcal{Y}[u_t - \gamma f - u_{xx} - u_{yy}] \big|_{(4.50)} &= 0, \\
    \mathcal{Y}[v_t - \gamma g - d(v_{xx} + v_{yy})] \big|_{(4.50)} &= 0, \\
    \mathcal{Y}[f_t] \big|_{(4.50)} &= 0, \\
    \mathcal{Y}[f_x] \big|_{(4.50)} &= 0, \\
    \mathcal{Y}[f_y] \big|_{(4.50)} &= 0, \\
    \mathcal{Y}[g_t] \big|_{(4.50)} &= 0, \\
    \mathcal{Y}[g_x] \big|_{(4.50)} &= 0, \\
    \mathcal{Y}[g_y] \big|_{(4.50)} &= 0
\end{align*}
\]

(4.52)

where \( \mathcal{Y} \) is the prolonged operator. Here \( \mathcal{Y} \) is written as

\[
\begin{align*}
    \mathcal{Y} &= Y + \zeta^1 \frac{\partial}{\partial u_t} + \zeta^2 \frac{\partial}{\partial u_v} + \zeta^3 \frac{\partial}{\partial u_{xx}} + \zeta^4 \frac{\partial}{\partial u_{yy}} + \zeta^5 \frac{\partial}{\partial v_t} + \zeta^6 \frac{\partial}{\partial v_v} + \zeta^7 \frac{\partial}{\partial v_{xx}} + \zeta^8 \frac{\partial}{\partial v_{yy}} + \\
    &+ \omega^1 \frac{\partial}{\partial f_t} + \omega^2 \frac{\partial}{\partial f_v} + \omega^3 \frac{\partial}{\partial f_f} + \omega^4 \frac{\partial}{\partial f_g} + \omega^5 \frac{\partial}{\partial g_t} + \omega^6 \frac{\partial}{\partial g_v} + \omega^7 \frac{\partial}{\partial g_f} + \omega^8 \frac{\partial}{\partial g_g}.
\end{align*}
\]

(4.53)
From the equations (4.52) it can be found after some calculations that, the Lie algebra \( L_\mathcal{E} \) is spanned by the nine operators

\[
Y_1 = v \frac{\partial}{\partial v} + g \frac{\partial}{\partial g}, \quad Y_2 = \frac{\partial}{\partial v}, \quad Y_3 = u \frac{\partial}{\partial u} + f \frac{\partial}{\partial f}, \quad Y_4 = \frac{\partial}{\partial u}, \quad \\
Y_5 = t \frac{\partial}{\partial t} + \frac{x}{2} \frac{\partial}{\partial x} + \frac{y}{2} \frac{\partial}{\partial y} - f \frac{\partial}{\partial f} - g \frac{\partial}{\partial g}, \quad Y_6 = \frac{\partial}{\partial t}, \quad \\
Y_7 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, \quad Y_8 = \frac{\partial}{\partial x}, \quad Y_9 = \frac{\partial}{\partial y}. 
\]

(4.54)

The commutator table for the equivalence algebra \( L_\mathcal{E} \) (4.54) is given by

<table>
<thead>
<tr>
<th>([Y_r, Y_c])</th>
<th>(Y_1)</th>
<th>(Y_2)</th>
<th>(Y_3)</th>
<th>(Y_4)</th>
<th>(Y_5)</th>
<th>(Y_6)</th>
<th>(Y_7)</th>
<th>(Y_8)</th>
<th>(Y_9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Y_1)</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(Y_2)</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(Y_3)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(Y_4)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(Y_5)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>(-\frac{1}{2} Y_8)</td>
<td>(-\frac{1}{2} Y_9)</td>
<td>0</td>
</tr>
<tr>
<td>(Y_6)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(Y_7)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(Y_8)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(\frac{1}{2} Y_8)</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(Y_9)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(\frac{1}{2} Y_9)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

If \( d = 1 \) in the equations (4.35a) then the Lie algebra \( L_\mathcal{E} \) (4.54) is extended with the two operators

\[
Y_{d_1} = v \frac{\partial}{\partial u} + g \frac{\partial}{\partial f}; \quad Y_{d_2} = u \frac{\partial}{\partial v} + f \frac{\partial}{\partial g}. 
\]

As mentioned in the section 4.1 the principal algebra \( L_\mathcal{P} \) which is a subalgebra of \( L_\mathcal{E} \), could be found from \( L_\mathcal{E} \). This can be done in the following way (see [4]). The general operator of \( L_\mathcal{E}, Y \) is given by

\[
Y = \lambda_1 Y_1 + \lambda_2 Y_2 + \lambda_3 Y_3 + \lambda_4 Y_4 + \lambda_5 Y_5 + \lambda_6 Y_6 + \lambda_7 Y_7 + \lambda_8 Y_8 + \lambda_9 Y_9. 
\]

The general operator \( X \) for the principal Lie algebra \( L_\mathcal{P} \) can be found from the condition

\[
X = \text{pr}_{(x,y,u,v)} Y 
\]

where \( Y \) satisfy \( \text{pr}_{(u,v,f,g)} Y = 0 \). Here \( \text{pr}_{(u,v,f,g)} Y \) is the projection of the operator \( Y \) to \((u, v, f, g)\) (see [3],[4]). From \( \text{pr}_{(u,v,f,g)} Y = 0 \) we have

\[
(\lambda_3 u + \lambda_4) \frac{\partial}{\partial u} + (\lambda_1 v + \lambda_2) \frac{\partial}{\partial v} + (\lambda_3 - \lambda_5) f \frac{\partial}{\partial f} + (\lambda_1 - \lambda_5) g \frac{\partial}{\partial g} = 0 
\]
from this we can conclude that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$. So we get

$$X = \text{pr}_{(t,x,y,u,v)} (\lambda_6 Y_6 + \lambda_7 Y_7 + \lambda_8 Y_8 + \lambda_9 Y_9)$$

and a basis for $L_P$ is made up of the four operators

$$X_6 = Y_6, \quad X_7 = Y_7, \quad X_8 = Y_8, \quad X_9 = Y_9. \quad (4.55)$$

### 4.8.2 Three space dimensions

Calculation of equivalence group for the equations (2.23) in three space dimensions, the equations are here written as

$$u_t = \gamma f(u, v) + u_{xx} + u_{yy} + u_{zz},$$
$$v_t = \gamma g(u, v) + d(v_{xx} + v_{yy} + v_{zz}) \quad (4.56a)$$

where $f(u, v)$ and $g(u, v)$ are two arbitrary functions. Also need for the calculation of the equivalence group is the equations

$$f_t = f_x = f_y = f_z = 0, \quad g_t = g_x = g_y = g_z = 0. \quad (4.56b)$$

According to the theory [1],[3]-[5]. To find the equivalence group $E_c$, we search for the equivalence operator $Y$ in the form

$$Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v} + \mu^1 \frac{\partial}{\partial f} + \mu^2 \frac{\partial}{\partial g} \quad (4.57)$$

where the functions $\xi^i$ and $\eta^j$ depend on the variables $(t, x, y, z, u, v)$. Also $\mu^k$ are functions of the variables $(t, x, y, z, u, v, f, g)$. The operator $Y$ after the necessary prolongation should satisfy the following invariance test (see [1]) on the equations (4.56)

$$\tilde{Y} [u_t - \gamma f - u_{xx} - u_{yy} - u_{zz}] \big|_{(4.56)} = 0,$$
$$\tilde{Y} [v_t - \gamma g - d(v_{xx} + v_{yy} + v_{zz})] \big|_{(4.56)} = 0,$$
$$\tilde{Y} [f_t] \big|_{(4.56)} = 0, \quad \tilde{Y} [f_x] \big|_{(4.56)} = 0,$$
$$\tilde{Y} [f_y] \big|_{(4.56)} = 0, \quad \tilde{Y} [f_z] \big|_{(4.56)} = 0,$$
$$\tilde{Y} [g_t] \big|_{(4.56)} = 0, \quad \tilde{Y} [g_x] \big|_{(4.56)} = 0,$$
$$\tilde{Y} [g_y] \big|_{(4.56)} = 0, \quad \tilde{Y} [g_z] \big|_{(4.56)} = 0. \quad (4.58)$$
where $\tilde{Y}$ is the prolonged operator. Here $\tilde{Y}$ is written as

$$
\tilde{Y} = Y + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial v_t} + \zeta_{22} \frac{\partial}{\partial u_{xx}} + \zeta_{33} \frac{\partial}{\partial u_{yy}} + \zeta_{33} \frac{\partial}{\partial v_{yy}} + \\
+ \zeta_{44} \frac{\partial}{\partial u_{zz}} + \omega_1 \frac{\partial}{\partial f_t} + \omega_2 \frac{\partial}{\partial f_x} + \omega_3 \frac{\partial}{\partial f_y} + \omega_4 \frac{\partial}{\partial f_z} + \\
+ \omega_2 \frac{\partial}{\partial g_t} + \omega_3 \frac{\partial}{\partial g_x} + \omega_4 \frac{\partial}{\partial g_y} + \omega_4 \frac{\partial}{\partial g_z}.
$$

(4.59)

From the equations (4.58) it can be found after some calculations that, the Lie algebra $L_E$ is spanned by the twelve operators

$$Y_1 = t \frac{\partial}{\partial t} + \frac{1}{2} x \frac{\partial}{\partial x} + \frac{1}{2} y \frac{\partial}{\partial y} + \frac{1}{2} z \frac{\partial}{\partial z} - f \frac{\partial}{\partial f} - g \frac{\partial}{\partial g},
$$

$$Y_2 = \frac{\partial}{\partial t}, \quad Y_3 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, \quad Y_4 = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z},
$$

$$Y_5 = z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z}, \quad Y_6 = \frac{\partial}{\partial x}, \quad Y_7 = \frac{\partial}{\partial y}, \quad Y_8 = \frac{\partial}{\partial z},
$$

$$Y_9 = u \frac{\partial}{\partial u} + f \frac{\partial}{\partial f}, \quad Y_{10} = v \frac{\partial}{\partial v} + g \frac{\partial}{\partial g}, \quad Y_{11} = \frac{\partial}{\partial u}, \quad Y_{12} = \frac{\partial}{\partial v}.
$$

(4.60)

If $d = 1$ in the equations (4.35a) then the Lie algebra $L_E$ (4.60) is extended with the two operators

$$Y_{d_1} = v \frac{\partial}{\partial u} + g \frac{\partial}{\partial f}, \quad Y_{d_2} = u \frac{\partial}{\partial v} + f \frac{\partial}{\partial g}.
$$

4.9 Admitted group for a bacterial pattern model

The equation system (2.32) in one-dimension with $k_3 = k_7 = k_8 = 0$ and $\chi(n, c) = k_1 n/(k_2 + c)^2$ can be written as

$$n_{tt} = D_n n_{xx} - \frac{k_1 n c_x}{(k_2 + c)^2} - \frac{k_1 n c_{xx}}{(k_2 + c)^2} + 2 \frac{k_1 n c^2_x}{(k_2 + c)^3},
$$

(4.61)

$$c_{tt} = D_c c_{xx} + \frac{k_5 s n^2}{(k_6 + n^2)},
$$

(4.62)

$$s_{tt} = D_s s_{xx}.
$$

(4.63)

The admitted group for the equations (4.61)-(4.63) is generated by the three operators

$$X_1 = -t \frac{\partial}{\partial t} - \frac{1}{2} x \frac{\partial}{\partial x} + s \frac{\partial}{\partial s}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = \frac{\partial}{\partial x}.
$$
4.10 Admitted group of a mechanochemical model

4.10.1 One space dimension

The equations (2.45) in one-dimensional can be written as

\[ n_t = D_1 n_{xx} - D_2 n_{xxxx} - a_1(n_x p_x + n p_{xx}) + a_2(n_x p_{xxx} + n p_{xxxx}) - \]
\[ - n_x u_t - n u_{tx} + r n (1 - n), \]  
\[ \tau n_x - \lambda \tau n^2 n_x (\rho + \gamma \rho_{xx}) + \frac{\tau n}{1 + \lambda n^2} (\rho + \gamma \rho_{xxx}), \]  
\[ \rho_t = - \rho_x u_t - \rho u_{tx} \quad \text{(4.66)} \]

where \( \mu = \mu^1 + \mu^2 \) and \( \tilde{\nu} = (1 + \nu') \). The admitted group for the equations (4.64)-(4.66) is generated by the two operators

\[ X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}. \]

4.10.2 Two space dimensions

In two space dimensions \((x, y)\) and with \( u = (u^1, u^2) \) the equations (2.45) are written as

\[ n_t = D_1 (n_{xx} + n_{yy}) - D_2 (n_{xxxx} + 2 n_{xxxy} + n_{yyyy}) - \]
\[ - a_1 (n_x \rho_x + n (\rho_{xx} + \rho_{yy}) + n_y \rho_y) + \]
\[ + a_2 (n_x (\rho_{xxx} + \rho_{xyy}) + n (\rho_{xxxx} + 2 \rho_{xxxy} + \rho_{yyyy}) + n_y (\rho_{xxy} + \rho_{yy})) - \]
\[ - n_x u^1_t - n (u^1_{tx} + u^1_{ty}) - n_y u^2_t + r n (1 - n), \]
\[ \tau n_x - \tau \lambda n^2 n_x (\rho + \gamma (\rho_{xx} + \rho_{yy})) + \frac{\tau n}{1 + \lambda n^2} (\rho + \gamma (\rho_{xxx} + \rho_{xyy})), \]  
\[ \tau n_y - \tau \lambda n^2 n_y (\rho + \gamma (\rho_{xx} + \rho_{yy})) + \frac{\tau n}{1 + \lambda n^2} (\rho + \gamma (\rho_{xxy} + \rho_{yy})), \]  
\[ \rho_t = - \rho_x u^1_t - \rho (u^1_{tx} + u^2_{ty}) - \rho_y u^2_t \quad \text{(4.67)} \]
The admitted group for the equations (4.67) is generated by the three operators

\[ X_1 = \frac{\partial}{\partial t}, \quad X_2 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} - u^2 \frac{\partial}{\partial u_1} + u^1 \frac{\partial}{\partial u_2}, \]

\[ X_3 = \frac{\partial}{\partial x}, \quad X_4 = \frac{\partial}{\partial y}. \] (4.68)

### 4.11 Equivalence group for an avascular tumour model

Calculating the continuous equivalence group \( \mathcal{E}_c \) for the equations (2.60). The equations are here written as

\[
\begin{align*}
n_t + v n_r - a n + b n^2 &= 0, \\
2 c_r c_r + c_{rr} - k n - \nu(c_t + v c_r + b c n) &= 0, \\
2 \frac{v}{r} + v_r - b n &= 0.
\end{align*}
\] (4.69a)

Here with the three arbitrary functions \( a(c), b(c) \) and \( k(c) \). Also needed is the following equations

\[
\begin{align*}
a_t &= 0, \quad a_r = 0, \quad a_n = 0, \quad a_v = 0 \quad b_t = 0, \quad b_r = 0, \quad b_n = 0, \quad b_v = 0, \\
b_t &= 0, \quad b_r = 0, \quad k_t = 0, \quad k_r = 0, \quad k_n = 0, \quad k_v = 0.
\end{align*}
\] (4.69b, 4.69c)

According to the theory [1],[3]-[5]. To find the equivalence group \( \mathcal{E}_c \), we search for the equivalence operator \( Y \) in the form

\[ Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial r} + \eta^1 \frac{\partial}{\partial n} + \eta^2 \frac{\partial}{\partial v} + \eta^3 \frac{\partial}{\partial c} + \mu^1 \frac{\partial}{\partial a} + \mu^2 \frac{\partial}{\partial b} + \mu^3 \frac{\partial}{\partial k}. \] (4.70)

with the unknown functions

\[ \xi^i = \xi^i(t, r, n, v, c), \quad \eta^j = \eta^j(t, r, n, v, c), \quad \mu^k = \mu^k(t, r, n, v, c, a, b, k). \]
The operator $Y$ after the necessary prolongation should satisfy the following invariance test (see [1]) on the equations (4.69):

$$
\left[ \zeta_1 + v \varsigma_2^1 + + n_r \eta^2 - a \eta^1 - n \mu^1 + n^2 \mu^2 + 2bn \eta^1 \right]_{(4.69)} = 0,
$$

$$
\left[ \left( - \frac{2c^r}{r^2} \varsigma^2 - n \xi^3 - n \varsigma_1^3 - \nu \varsigma_1^3 \right) \eta^1 + (k + \nu bc) \varsigma_2^3 \right]_{(4.69)} = 0,
$$

$$
\left[ - \frac{2v}{r^2} \varsigma^2 + 2 \eta^2 - n \mu^2 - b \eta^1 + \varsigma_2^2 \right]_{(4.69)} = 0, \quad (4.71)
$$

From the equations (4.71) it can be found after some calculations that, the continuous equivalence group $E_c$ is generated by the twelve operators:

$$
\begin{align*}
Y_1 &= \frac{\partial}{\partial t}, \\
Y_2 &= c \frac{\partial}{\partial c} + k \frac{\partial}{\partial k}, \\
Y_3 &= \frac{\partial}{\partial c} - \nu b \frac{\partial}{\partial k}, \\
Y_4 &= -n \frac{\partial}{\partial n} + b \frac{\partial}{\partial b} + k \frac{\partial}{\partial k}, \\
Y_5 &= t \frac{\partial}{\partial t} + \frac{r}{2} \frac{\partial}{\partial r} - n \frac{\partial}{\partial n} - \frac{v}{2} \frac{\partial}{\partial v} - a \frac{\partial}{\partial a}.
\end{align*}
$$

(4.73)

Projection on operators (see [3],[4]) are done according to the following rules

$$
X_i = \text{pr}_{(t,r,n,c,v)} Y_i, \quad Z_i = \text{pr}_{(c,a,b,k)} Y_i.
$$

The principal Lie algebra $L_P$ is found to be spanned by the operator

$$
X_1 = \frac{\partial}{\partial t}. \quad (4.74)
$$
4.11.1 Extension of principal Lie algebra with $Y_6$

Taking an operator $Y_6$ so that $Y_1$ and $Y_6$ span a Lie algebra (subalgebra of $L_ε$). Here taking $Y_6$ in the form

$$Y_6 = \lambda_3 Y_3 + \lambda_4 Y_4 + \lambda_5 Y_5 = \lambda_5 \frac{\partial}{\partial t} + \frac{\lambda_5}{2} \frac{r}{\partial r} - (\lambda_4 + \lambda_5) n \frac{\partial}{\partial n} + \lambda_3 \frac{\partial}{\partial c} - \frac{\lambda_5}{2} v \frac{\partial}{\partial v}$$

where $\lambda_3, \lambda_4$ and $\lambda_5$ are constants. Then the question is how to find those equations from (4.69a) that admits the operators $X_1$ and $X_6$.

$$X_1 = \frac{\partial}{\partial t}, \quad X_6 = \lambda_5 t \frac{\partial}{\partial t} + \frac{\lambda_5}{2} r \frac{\partial}{\partial r} - (\lambda_4 + \lambda_5) n \frac{\partial}{\partial n} + \lambda_3 \frac{\partial}{\partial c} - \frac{\lambda_5}{2} v \frac{\partial}{\partial v}. \quad (4.75)$$

This can be done using the theory in [3],[4], by first finding the equations

$$a = A(c), \quad b = B(c), \quad k = K(c) \quad (4.76)$$

which are invariant with respect to $Z_6$. In this case we find

$$a = C_1 e^{-\lambda_2 c}, \quad b = C_2 e^{\lambda_3 c}, \quad k = (C_3 - \nu C_2 c) e^{\lambda_4 c} \quad (4.77)$$

where $C_1, C_2$ and $C_3$ are arbitrary constants. Then the equations (4.69a) after the substitutions (4.77) are written as

$$n_t + vn_r - C_1 e^{-\lambda_2 c} n + C_2 e^{\lambda_3 c} n^2 = 0,$$

$$2C_r + c_{rr} - (C_3 - \nu C_2 c) e^{\lambda_4 c} n - \nu (c_t + v c_r + C_2 \frac{\lambda_4 c}{\lambda_3} c n) = 0, \quad (4.78)$$

$$2v_r + v_r - C_2 e^{\lambda_3 c} n = 0.$$

So from the equivalence algebra $L_ε$, this are the equations that admits the operators $X_1$ and $X_6$.

4.11.2 Extension of principal Lie algebra with $Y_7$

Taking an operator $Y_7$ so that $Y_1$ and $Y_7$ span a Lie algebra (subalgebra of $L_ε$). Here taking $Y_7$ in the form $Y_7 = \lambda_2 Y_2 + \lambda_3 Y_3 + \lambda_4 Y_4 + \lambda_5 Y_5$

$$Y_7 = \lambda_5 t \frac{\partial}{\partial t} + \frac{\lambda_5}{2} \frac{r}{\partial r} - (\lambda_4 + \lambda_5) n \frac{\partial}{\partial n} + (\lambda_2 c + \lambda_3) \frac{\partial}{\partial c} - \frac{\lambda_5}{2} v \frac{\partial}{\partial v} - \lambda_5 a \frac{\partial}{\partial a} + \lambda_4 b \frac{\partial}{\partial b} + (\lambda_2 k + \lambda_4 k - \lambda_3 \nu b) \frac{\partial}{\partial k}$$

where $\lambda_2, \lambda_3$ and $\lambda_4$ are constants.
where $\lambda_2, \lambda_3, \lambda_4$ and $\lambda_5$ are constants. To find the equations from (4.69a) that admits the operator

$$X_7 = \lambda_5 t \frac{\partial}{\partial t} + \frac{\lambda_5}{2} r \frac{\partial}{\partial r} - (\lambda_4 + \lambda_5) n \frac{\partial}{\partial n} + (\lambda_2 c + \lambda_3) \frac{\partial}{\partial c} - \frac{\lambda_5}{2} v \frac{\partial}{\partial v}. \quad (4.79)$$

Using the theory from [3],[4] leads us to investigate the conditions for the following equations

$$a = A(c), \quad b = B(c), \quad k = K(c) \quad (4.80)$$

to be invariant with respect to $Z_7$. Here it is found that

$$a = C_1 (\lambda_2 c + \lambda_3)^{\frac{\lambda_2}{\lambda_5}}, \quad b = C_2 (\lambda_2 c + \lambda_3)^{\frac{\lambda_2}{\lambda_5}},$$

$$k = \left( \frac{C_2 \lambda_3 \nu + (\lambda_2 c + \lambda_3) \lambda_2 C_3}{\lambda_2} \right) (\lambda_2 c + \lambda_3)^{\frac{\lambda_2}{\lambda_5}} \quad (4.81)$$

where $C_1, C_2$ and $C_3$ are arbitrary constants. Then the equations (4.69a) after the substitutions (4.81) are written as

$$n_t + v n_r - C_1 (\lambda_2 c + \lambda_3)^{\frac{\lambda_2}{\lambda_5}} n + C_2 (\lambda_2 c + \lambda_3)^{\frac{\lambda_2}{\lambda_5}} n^2 = 0,$$

$$2 \frac{c_r}{r} + c_{rr} - \left( \frac{C_2 \lambda_3 \nu + (\lambda_2 c + \lambda_3) \lambda_2 C_3}{\lambda_2} \right) (\lambda_2 c + \lambda_3)^{\frac{\lambda_2}{\lambda_5}} n -$$

$$-\nu (c_t + v c_r + C_2 (\lambda_2 c + \lambda_3)^{\frac{\lambda_2}{\lambda_5}} c n) = 0,$$

$$2 \frac{v}{r} + v_r - C_2 (\lambda_2 c + \lambda_3)^{\frac{\lambda_2}{\lambda_5}} n = 0. \quad (4.82)$$

So from the equivalence algebra $L_E$, this are the equations that admits the operators $X_1$ and $X_7$. 
III Solutions

5 Invariant solutions

5.1 Introduction

If a system of differential equations admits a group $G$ then it is possible to try and find invariant solutions for this system, see [1].

As an concrete example consider the system of equations

$$
\begin{align*}
u_t &= F \left( \frac{v}{u + K} \right) \frac{\gamma}{u + K} + u_{xx}, \\
v_t &= G \left( \frac{v}{u + K} \right) \frac{\gamma}{u + K} + d v_{xx}
\end{align*}
$$

(5.1)

where $d$, $\gamma$ and $K$ are constants, $F$ and $G$ are arbitrary functions. The system (5.1) admits the group generated by the operator

$$
X = 2t \frac{\partial}{\partial t} + (x + 1) \frac{\partial}{\partial x} + (u + K) \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}.
$$

Invariant solutions for the system (5.1) could be found in the following way. Following the theory (see [1]), the invariants $J(t, x, u, v)$ under $X$ can be found from the equation

$$
X[J(t, x, u, v)] = 0.
$$

A possible basis of invariants for the operator $X$ are given by

$$
J_1 = \frac{x + 1}{\sqrt{t}}, \quad J_2 = \frac{u + 1}{\sqrt{t}}, \quad J_3 = \frac{v}{\sqrt{t}}.
$$

Then setting

$$
\theta = J_1, \quad \psi_1 = J_2, \quad \psi_2 = J_3
$$

and according to the theory (see [1]) $\psi_1$ and $\psi_2$ are taken as functions of $\theta$. Solving for $u$ and $v$ gives the form of $u$ and $v$ that is used when looking for the invariant solutions. Here $u$ and $v$ takes the form

$$
u = \psi_1(\theta) \sqrt{t} - K, \quad v = \psi_2(\theta) \sqrt{t}.
$$

(5.2)
By substituting the expressions for \(u\) and \(v\) (5.2) into the system (5.1) the system is reduced to the following ordinary differential equation in the variables \(\theta, \psi_1\) and \(\psi_2\):
\[
\begin{align*}
2\psi_1'' + \theta\psi_1' - \psi_1 + 2\frac{\gamma}{\psi_1}F\left(\frac{\psi_2}{\psi_1}\right) &= 0, \\
2d\psi_2'' + \theta\psi_2' - \psi_2 + 2\frac{\gamma}{\psi_1}G\left(\frac{\psi_2}{\psi_1}\right) &= 0.
\end{align*}
\] (5.3)

Once the solution of \(\psi_1(\theta)\) and \(\psi_2(\theta)\) are found from the equations (5.3), then the invariant solutions for the system (5.1) are obtained from (5.2).

5.2 Invariant solutions for the equations (4.9)

The equations (4.9) with the specific form for \(f(u)\) defined by (4.16) is given by the equations (4.17). In section 4.3.1 it is shown that the equations (4.17) admit the operator \(X_5\) in (4.14). The invariants \(J(t, x, u, v)\) under the operator \(X_5\) are found from the equation

\[X_5[J(t, x, u, v)] = 0.\]

The characteristic system for this equation is written as
\[dt = \frac{du}{\gamma u + \lambda_4} = \frac{dv}{\gamma^2 v + \lambda_4 b}\]
from which the following three invariants \(\theta, \psi_1, \psi_2\) are found
\[
\theta = x, \quad \psi_1 = \frac{(\gamma u + \lambda_4)e^{-\gamma t}}{\gamma}, \quad \psi_2 = \frac{(\gamma^2 v + \lambda_4 b)e^{-\gamma t}}{\gamma^2}.
\]

With \(\psi_1\) and \(\psi_2\) as functions of \(\theta\), we look for the invariant solutions in the form of
\[
u = e^{\gamma t}\psi_1(\theta) - \frac{\lambda_4}{\gamma}, \quad v = e^{\gamma t}\psi_2(\theta) - \frac{\lambda_4}{\gamma^2 b}.
\] (5.4)

Substituting the expressions for \(u\) and \(v\) (5.4) into the equations (4.17) gives
\[
-\gamma K_1\psi_1 + \psi_2 - D\psi_1'' = 0, \tag{5.5a}
\]
\[
\psi_2 = \frac{b}{2\gamma} \psi_1. \tag{5.5b}
\]

Substituting (5.5b) into equation (5.5a) yields
\[
\left(\frac{b}{2\gamma} - \gamma K_1\right)\psi_1 - D\psi_1'' = 0, \tag{5.6}
\]
a linear ordinary differential equation with constant coefficients. Solving this equation with the requirement \((b - 2\gamma^2 K_1)/(D\gamma) > 0\) gives
\[
\psi_1(\theta) = C_1 e^{\frac{1}{2\gamma^2}(b-2\gamma^2 K_1)\theta} + C_2 e^{-\frac{1}{2\gamma^2}(b-2\gamma^2 K_1)\theta}
\]
(5.7)
where \(C_1\) and \(C_2\) are arbitrary constants. Using (5.5b) we have
\[
\psi_2(\theta) = \frac{b}{2\gamma} \left[ C_1 e^{\frac{1}{2\gamma^2}(b-2\gamma^2 K_1)\theta} + C_2 e^{-\frac{1}{2\gamma^2}(b-2\gamma^2 K_1)\theta} \right].
\]
(5.8)
From (5.4) the invariant solutions are now written as
\[
u(t, x) = e^{\gamma t} \left[ C_1 e^{\frac{1}{2\gamma^2}(b-2\gamma^2 K_1)x} + C_2 e^{-\frac{1}{2\gamma^2}(b-2\gamma^2 K_1)x} \right] - \frac{\lambda_4}{\gamma}.
\]
(5.9)

### 5.3 Invariant solutions for the equations (4.18)

In section 4.4.2 the equations (4.31), a special case of equations (4.18a), are found to admit along with the principal algebra (4.23) also the operator
\[
X_{10} = \lambda_2 \frac{\partial}{\partial t} + n \frac{\partial}{\partial n} + c \frac{\partial}{\partial c}
\]
where \(\lambda_2\) is a constant. Consider the equations (4.31) with the arbitrary functions \(F_1, F_2\) and \(F_3\) in (4.30) given by
\[
F_1(c/n) = 0, \quad F_2(c/n) = c/n, \quad F_3(c/n) = C_2
\]
where \(C_2\) is a constant. Then the functions \(f, g, h\) and \(d\) in (4.30) are written as
\[
f = 0, \quad g = c, \quad h = C_2 n, \quad d = C_1.
\]
Using this function the equations (4.31) are now written as
\[
n_t - C_1 n_{xx} - C_2 n = 0, \quad c_t - D_c c_{xx} - c = 0.
\]
(5.10)

The invariants \(J(t, x, n, c)\) under the operator \(X_{10}\) are found from the equation
\[
X_{10}[J(t, x, n, c)] = 0.
\]
The characteristic system for this equation is written as
\[
\frac{dt}{\lambda_2} = \frac{dn}{n} = \frac{dc}{c}.
\]
from which the following three invariants $\theta, \psi_1, \psi_2$ are found

$$\theta = x, \quad \psi_1 = ne^{-(t/\lambda_2)}, \quad \psi_2 = ce^{-(t/\lambda_2)}.$$ 

Taking the invariant solutions with $\psi_1$ and $\psi_2$ as functions of $\theta$, solving for $n$ and $c$. Then the form of the invariant solutions are given by

$$n = e^{(t/\lambda_2)}\psi_1(\theta), \quad c = e^{(t/\lambda_2)}\psi_2(\theta). \quad (5.11)$$

Substituting the expressions for $n$ and $c$ in (5.11) into the equations (5.10) gives

$$\frac{C_2\lambda_2 - 1}{\lambda_2} \psi_1 + C_1\psi_1'' = 0, \quad (5.12a)$$
$$\frac{\lambda_2 - 1}{\lambda_2} \psi_2 + D_c\psi_2'' = 0. \quad (5.12b)$$

Solving this two equations with the requirements: $\lambda_2 C_1 > 0$, $\lambda_2 C_2 > 0$, $\lambda_2 D_c > 0$ and $\lambda_2 - 1 > 0$ gives

$$\psi_1(\theta) = K_1 \sin \left( \frac{\theta \sqrt{C_2\lambda_2 - 1}}{\sqrt{C_1\lambda_2}} \right) + K_2 \cos \left( \frac{\theta \sqrt{C_2\lambda_2 - 1}}{\sqrt{C_1\lambda_2}} \right),$$
$$\psi_2(\theta) = K_3 \sin \left( \frac{\theta \sqrt{C_2\lambda_2 - 1}}{\sqrt{C_1\lambda_2}} \right) + K_4 \cos \left( \frac{\theta \sqrt{C_2\lambda_2 - 1}}{\sqrt{C_1\lambda_2}} \right), \quad (5.13)$$

where $K_1, K_2, K_3$ and $K_4$ are arbitrary constants. From (5.11) the invariant solutions are now written as

$$n(t, x) = e^{(t/\lambda_2)} \left[ K_1 \sin \left( \frac{x \sqrt{C_2\lambda_2 - 1}}{\sqrt{C_1\lambda_2}} \right) + K_2 \cos \left( \frac{x \sqrt{C_2\lambda_2 - 1}}{\sqrt{C_1\lambda_2}} \right) \right],$$
$$c(t, x) = e^{(t/\lambda_2)} \left[ K_3 \sin \left( \frac{x \sqrt{C_2\lambda_2 - 1}}{\sqrt{C_1\lambda_2}} \right) + K_4 \cos \left( \frac{x \sqrt{C_2\lambda_2 - 1}}{\sqrt{C_1\lambda_2}} \right) \right]. \quad (5.14)$$

### 5.4 Invariant solutions for the equations (4.35)

In section 4.6.1 the equations (4.49), a special case of equations (4.35), are found to admit along with the principal algebra (4.41) also the operator

$$X_9 = \frac{\partial}{\partial t} + (u + K) \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}$$

where $K$ is a constant. Consider the equations (4.49) with the arbitrary functions $f$ and $g$ in (4.48) given by

$$f(u, v) = v + K_2 u + K_2 K, \quad g(u, v) = v$$
where $K_2$ is a constant. Then the equations (4.49) are written as

\begin{align*}
u_t &= \gamma(v + K_2u + K_2K) + u_{xx}, \\
v_t &= \gamma v + dv_{xx}. \tag{5.15}
\end{align*}

The invariants $J(t, x, u, v)$ under the operator $X_9$ are found from the equation

\[ X_9[J(t, x, u, v)] = 0. \]

The characteristic system for this equation is written as

\[ dt = \frac{du}{u + K} = \frac{dv}{v}, \]

from which the following three invariants $\theta, \psi_1, \psi_2$ are found

\[ \theta = x, \quad \psi_1 = (u + K)e^{-\gamma}, \quad \psi_2 = ve^{-\gamma}. \]

With $\psi_1$ and $\psi_2$ as functions of $\theta$, the invariant solutions are taken in the following form

\begin{align*}
u &= e^\theta \psi_1(\theta) - K, \\
\nu &= e^\theta \psi_2(\theta). \tag{5.16}
\end{align*}

Substituting the expressions for $\nu$ and $v$ in (5.16) into the equations (5.15) gives

\begin{align*}
\psi_1'' + \gamma \psi_2 - (1 - \gamma K_2) \psi_1 &= 0, \tag{5.17a} \\
\psi_2'' - (1 - \gamma) \psi_2 &= 0. \tag{5.17b}
\end{align*}

Solving (5.17b) with the requirement $(1 - \gamma)/d > 0$ gives

\[ \psi_2(\theta) = C_1e^{(\sqrt{1-\gamma}\theta)/\sqrt{d}} + C_2e^{-(\sqrt{1-\gamma}\theta)/\sqrt{d}} \tag{5.18} \]

where $C_1$ and $C_2$ are arbitrary constants. Solving (5.17a) with the requirement $\gamma K_2 - 1 > 0$ gives

\[ \psi_1(\theta) = C_3 \sin \left( \theta \sqrt{\gamma K_2 - 1} \right) + C_4 \cos \left( \theta \sqrt{\gamma K_2 - 1} \right) - \frac{d\gamma}{1 - \gamma - d + d\gamma K_2} \left( C_1e^{(\sqrt{1-\gamma}\theta)/\sqrt{d}} + C_2e^{-(\sqrt{1-\gamma}\theta)/\sqrt{d}} \right) \tag{5.19} \]

where $C_3$ and $C_4$ are arbitrary constants. From (5.16) the invariant solutions are now written as

\begin{align*}
u(t, x) &= e^\theta \left[ C_3 \sin \left( x \sqrt{\gamma K_2 - 1} \right) + C_4 \cos \left( x \sqrt{\gamma K_2 - 1} \right) \right] - \\
&\quad - e^\theta \left[ \frac{d\gamma}{1 - \gamma - d + d\gamma K_2} \left( C_1e^{(\sqrt{1-\gamma}\theta)/\sqrt{d}} + C_2e^{-(\sqrt{1-\gamma}\theta)/\sqrt{d}} \right) \right] - K, \tag{5.20} \\
v(t, x) &= e^\theta \left[ C_1e^{(\sqrt{1-\gamma}\theta)/\sqrt{d}} + C_2e^{-(\sqrt{1-\gamma}\theta)/\sqrt{d}} \right].
\end{align*}
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Primary bibliography


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Secondary bibliography


Dictionary

dermal  of or relating to the skin or dermis.

dermis  the deep vascular inner layer of the skin. The dermis lies below the epidermis and contains a number of structures including blood vessels, nerves, hair follicles, smooth muscle, glands and lymphatic tissue. It is made up of dense connective tissue - collagen, elastin and reticular fibres are present. The main cell types are fibroblasts, adipocytes (fat storage) and macrophages.

epidermal  of, relating to, or arising from the epidermis.

epidermis  outer layer of the skin covering the exterior body surface of vertebrates. The outermost epidermis is made up of stratified squamous epithelium with an underlying basement membrane. It contains no blood vessels, and is nourished by diffusion from the dermis.

epidermal cell  any of the cells making up the epidermis

tissue  biological tissue is a group of cells with intercellular material that together perform a specific function. There are four basic types of tissue in the human body. These compose all the organs, structures and other contents. Epithelium - Lines, covers, protects, absorbs and secretes. Connective tissue - As the name suggests, connective tissue holds everything together. Blood is considered a connective tissue. Muscle tissue - Muscle cells contain contractile filaments that move past each other and change the size of the cell. Nervous tissue - cells forming the brain, spinal cord and peripheral nervous system.

embryology  the branch of biology that studies the formation and development of embryo from fertilization until birth.

embryogenesis  the study of the formation and development of the embryo.

morphogenesis  part of embryology, morphogenesis is concerned with the shapes of tissues, organs and entire organisms and the positions of the various specialized cell types.

fibroblast  A fibroblast is a cell that makes the structural fibers and ground substance of connective tissue. Fibroblasts can give rise to other cells, such as bone cells, fat cells, and smooth muscle cells. Note that all of these cells are of mesodermal origin.
mesenchyme  embryonic tissue of mesodermal origin, that forms connective tissue and blood and smooth muscles.

epithelium  a membranous cellular tissue that covers a free surface or lines a tube or cavity of an animal body and serves especially to enclose and protect the other parts of the body, to produce secretions and excretions, and to function in assimilation.

wound healing  wound healing is divided into inflammation, wound closer and extra-cellular matrix remodelling in scar tissue.

mitosis  is the process of chromosome segregation and nuclear division that follows replication of the genetic material in eukaryotic cells. Could also mean cell division in which mitosis occurs.

metabolism  is the uptake and digestion of food, and the disposal of waste products in living organisms. Because this process can happen at many levels within an organism, we can identify several kinds of metabolism: concerning an organism in its entirety (total metabolism), concerning a particular substance (specific metabolism), concerning a particular living cell (cell metabolism).

actin  is a contractile protein filament, important for cell movements. It is expressed in all body cells, but especially in muscle cells. Actin can polymerize into microfilament, which are essential for the cytoskeleton, for cell motility, and for contraction of the cell during cell division.

cytosol  The cytosol (as opposed to cytoplasm, which also includes the organelles) is the internal fluid of the cell, and a large part of cell metabolism occurs here. Contains a large number of different enzymes.

cytoplasm  is the viscid, semifluid matter contained within the plasma membrane of a cell. In contrast to the protoplasm, however, the cytoplasm does not include the cell nucleus. The watery or aqueous component of the cytoplasm is the cytosol, which includes ions and soluble macromolecules, for example enzymes. The insoluble constituents of the cytoplasm include the organelles and the cytoskeleton, which gives to the cytoplasm a gel like structure and consistency.

cytogel  is the interior of the cell.

cytoskeleton  is a cellular ”scaffolding” or ”skeleton”, used to maintain and/or alter cellular shape. The cytoskeleton is composed of actin filaments, microtubules, intermediate filaments, and other proteins.

protoplasm  is the substance inside the membrane of a living cell. At the simplest level, it is divisible into cytoplasm and the nucleus.
cell membrane  A component of every biological cell, it is a thin and structured layer of lipid and protein molecules, which surrounds the cell. It separates a cell’s interior from its surroundings and controls what moves in and out. Cell surface membranes often contain receptor proteins and cell adhesion proteins. These membrane proteins are important for the regulation of cell behavior and the organization of cells in tissues.

plasma membrane  see cell membrane

amoeboids  are cells that move or feed by means of temporary projections, called pseudopods or false feet. They have appeared among a number of different groups. Some cells in animals may be amoeboïd, such as white blood cells.

extracellular  beyond or outside the cell.

proliferation  the reproduction or multiplication of similar forms, especially of cells and morbid cysts.

blood vessel  All the vessels lined with endothelium through which blood circulates.

vascular  an adjective that describes tissue that is heavily endowed with blood vessels. Vascular also means ”relating to blood vessels”.

vasculogenesis  is the formation of (major) blood vessels by cells (endothelial cells and angioblasts). Blood vessels initially form throughout the entire embryo as endoderm induces the overlying splanchnopleuric mesoderm to form networks of vasculature characteristic of each specific region.

vascularisation  growth of blood vessels into a tissue with the result that the oxygen and nutrient supply is improved.

angiogenesis  the mechanism whereby preexisting blood vessels lengthen or branch by sprouting.

angioblast  a cell in the embryo which develops into blood vessel tissue. An embryonic mesenchymal tissue which differentiates into the blood cells and blood vessels.

endothelium  the layer of epithelial cells that lines the cavities of the heart and of the blood and lymph vessels and the serous cavities of the body, originating from the mesoderm.

endothelial cells  these cells arise from angioblasts to form the initial vascular network. They provide the endothelial lining for the entire cardiovascular system.

neoplasm  New and abnormal growth of tissue, which may be benign or cancerous.
**neoplastic**  Pertaining to or like a neoplasm with new and abnormal growth.

**tumour**  An abnormal mass of tissue that results from excessive cell division that is uncontrolled and progressive, also called a neoplasm.

**solid tumour**  A cancer that originates in organ or tissue other than bone marrow or the lymph system.

**tumour angiogenesis factor**  Substance released from a tumour that promotes vascularisation of the mass of neoplastic cells. Once a tumour becomes vascularised, it will grow more rapidly and is more likely to metastasise.

**avascular**  without blood or lymphatic vessels

**benign tumour**  A nonmalignant clone of neoplastic cells that does not invade locally or spread to other parts of the body (metastasise), having lost growth control but not positional control.

**malignant tumour**  a mass of cancer cells. These cells have uncontrolled growth and will invade surrounding tissues and spread to distant sites of the body, setting up new cancer sites, a process called metastasis.

**hypoxia**  reduction of oxygen supply to tissue below physiological levels despite adequate perfusion of the tissue by blood.

**metastasis**  the transfer of disease from one organ or part to another not directly connected with it. See also malignant tumour.

**neuron**  neurons are the primary cells of the nervous system.

**synapse**  neurons join to one another and to other cells through synapses, which connect the axon tip of one cell to a dendrite of another, or less commonly to its axon or soma.

**gastrula**  an early metazoan embryo in which the ectoderm, mesoderm, and endoderm are established.

**gastrulation**  The process in which a gastrula develops from a blastula by the inward migration of cells.

**germ layer**  (embryology) any of the 3 layers of cells differentiated in embryos following gastrulation. Outer layer is called ectoblast, ectoderm or exoderm. Middle layer is called mesoblast or mesoderm. The inner layer is called endoblast, endoderm, entoblast, entoderm or hypoblast.
**ectoblast, ectoderm, exoderm** the outer germ layer that develops into skin and nervous tissue.

**mesoblast, mesoderm** the middle germ layer that develops into muscle and bone and cartilage and blood and connective tissue.

**endoblast, endoderm, entoblast, entoderm, hypoblast** the inner germ layer that develops into the lining of the digestive and respiratory systems.

**protein** is a complex, high molecular weight organic compound that consists of amino acids joined by peptide bonds. Proteins are essential to the structure and function of all living cells and viruses.

**enzyme** An enzyme is a protein, or protein complex, that catalyzes a chemical reaction in an organism. Enzymes react selectively on definite compounds called substrates. The substrate attach to the enzyme at what is called a site for the enzyme. Catalysis takes place at a particular site on the enzyme called the active site. Enzymes are very specific as to the reactions they catalyze and the chemicals (substrates) that are involved in the reactions. Substrates fit their enzymes like a key fits its lock.

**taxis** In biology: The responsive movement of a free-moving organism or cell toward or away from an external stimulus, such as light.

**chemotaxis** A response of motile cells or organisms in which the direction of movement is affected by the gradient of a diffusible substance. Differs from chemokinesis in that the gradient alters probability of motion in one direction only, rather than rate or frequency of random motion.

**haptotaxis** Strictly speaking, a directed response of cells in a gradient of adhesion, but often loosely applied to situations where an adhesion gradient is thought to exist and local trapping of cells seems to occur.

**galvanotaxis** The directed movement of cells induced by an applied voltage.

**in vitro** outside the living body and in an artificial environment.

**in vivo** in the living body of a plant or animal.

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Abstract. Recently, several mathematical models appeared in the literature for describing spread of malignant tumours. These models are formulated as systems of nonlinear partial differential equations containing, in general, several unknown functions of dependent variables. Determination of these unknown functions is a complicated problem that challenges researchers. In this master thesis, three different models (originated from one model) are considered. The generators of the equivalence groups and invariant solutions are calculated for these three models.

Introduction

For the past years, mathematical modelling in medicine and biology has become a growing field in applied mathematics. One of the active research areas are dealing with mathematical models, describes tumour growth. The aim of studying these kind of models is to understand the mechanisms that may control the development of tumours, and reduce the necessity of time-consuming and complicated experiments. Hopefully, mathematical modelling of tumour growth will give some useful results to develop new treatments and improve the prognosis of patients diagnosed with cancer.

Tumours are classified as either benign or malignant. The main feature that distinguishes benign from malignant tumours is that benign tumours are encapsulated by a layer of collagen, the chemical basis of ordinary connective tissue, while malignant tumours has the ability to invade surrounding tissue or colonize other organs in the body via the circulatory systems (metastasis).

The development of solid tumours can be divided into two distinct phases of growth. In the initial avascular phase, the tumour receives vital nutrients (such as oxygen and glucose) and eliminates waste products via diffusive transport. However, diffusion alone can not provide any larger tumour with vital nutrients and, in the absence of blood supply, the size of the tumour will be limited. During the avascular phase, the tumour may be considered as spherical with a hypoxic (low oxygen) core of dormant or dead cells surrounded by a layer of proliferating cells.
To survive, the hypoxic tumour cells secrete a substance, so called tumour angiogenesis factor (TAF), that stimulate neighbouring blood vessels to grow towards the tumour. The process causing proliferation of new blood vessels is called angiogenesis. Angiogenesis is not unique to tumour growth, the angiogenesis factor is also released by hypoxic macrophages to initiate revascularization (restoration of blood supply) in wound healing. A network of blood vessels will be developed which eventually penetrate the tumour and supply it with vital nutrients from the circulatory system. When the tumour is furnished with a circulatory blood supply it starts to grow rapidly. The tumour has reached the second vascular phase, with an almost unlimited supply of nutrients.

**Mathematical model**

Invasion of malignant cells into neighbouring tissue follows from an invasive constitution (phenotype), characterized by the process of adhesion, local proteolysis (i.e. digestion of proteins by cellular enzymes called proteases) and migration.

Perumpanani et al. [1] formulated a mathematical model describing the dynamical interplay between the directed movement of invasive cells in response to a fixed substrate (haptotaxis) and proteolysis. Ignoring cellular diffusion, they derived a model based on a continuous approach in which $u(t, x)$, $c(t, x)$ and $p(t, x)$ represent the concentrations of invasive cells, cellular matrix (proteins between cells, such as collagen) and protease. The model is written in terms of a system of partial differential equations:

\[
\frac{\partial u}{\partial t} = f(u) - k_3 \frac{\partial}{\partial x} \left[ u \frac{\partial c}{\partial x} \right] \\
\frac{\partial c}{\partial t} = -g(c, p) \\
\frac{\partial p}{\partial t} = h(u, c) - Kp,
\]

where $t$ and $x$ are time and space variables. Here, the functions $f(u)$, $g(c, p)$ and $h(u, c)$ corresponds to the invasive cell proliferation, proteolysis and protease production, respectively. The term $k_3 \partial / \partial x (u \partial c / \partial x)$ describe the invasive cell movement by haptotaxis, $Kp$ is the natural decay of protease.

In chapter 1, a simplified model of the above system is considered, generators of the equivalence group for the model and group invariant solutions are calculated. In the following chapter, a general model similar to the system (0.1) is treated. Finally, a generalized model, originated from the system (0.1), being investigated (chapter 3).
1 The equivalence group and invariant solutions of a tumour growth model (simplified)

1.1 Mathematical model

Perumpanani et al. [1] investigated the problem of malignant cells into surrounding tissue neglecting cellular diffusion. Motivated by several important observations in tumour biology (e.g. [2]), they suggested a mathematical model for invasion by haptotaxis (i.e. directed movement that occur in response to a fixed substrate) and protease production. The model studies the averaged one-dimensional spatial dynamics of malignant cells by ignoring variations in the plane perpendicular to the direction of invasion. The model is formulated in terms of nonlinear partial differential equations as the following system:

\[
\begin{align*}
    u_t &= f(u) - (uc_x)_x \\
    c_t &= -g(c, p) \\
    p_t &= h(u, c) - Kp.
\end{align*}
\]

Here \( u, c \) and \( p \), depends on time \( t \) and one space coordinate \( x \) and represent the concentrations of invasive cells, extracellular matrix (e.g. type IV collagen) and protease, respectively. To describe the dynamics of a specific biological system, the authors of the model introduced arbitrary elements \( f(u), g(c, p) \) and \( h(u, c) \) that are supposed to be increasing functions of the dependent variables \( u, c, p \). For example, the function \( h(u, c) \) in last equation of the above system represents the dependence of the protease production on local concentrations of malignant cells and collagen, while the term \(-Kp\) is based on the assumption that the protease decays linearly, where \( K \) is a positive constant to be determined experimentally via half-life.

By observing that the timescales associated with the protease production and decay are considerably shorter than for the invading cells, the above model can be reduced to the following system of two equations [1, 3]:

\[
\begin{align*}
    u_t &= f(u) - (uc_x)_x \\
    c_t &= -g(c, u),
\end{align*}
\]

where \( f(u) \) and \( g(c, u) \) are arbitrary functions satisfying the conditions

\[ f(u) > 0, \quad g_c(c, u) > 0, \quad g_u(c, u) > 0. \]

The main part of the paper [1] is dedicated to discussion of the particular case \( f(u) = u(u - 1) \), corresponding to a logistic production rate, and \( g(c, u) = uc^2 \). Another particular case, namely \( g(c, u) = uh(c) \), involving two arbitrary functions \( f(u) \) and \( h(c) \) of one variable, is discussed in detail in [3], where the functions \( f(u) \) and \( h(c) \) are classified according to symmetries of the corresponding system (1.2).
Our goal is to find and employ the Lie algebra of the generators of the equivalence transformations for the general model (1.2) with arbitrary functions \( f(u) \) and \( g(c, u) \). Note that Eq. (1.2) do not involve explicitly the independent variables \( t \) and \( x \). Consequently, the system (1.2) with arbitrary functions \( f(u) \) and \( g(c, u) \) is invariant under the two-parameter group of translations of \( t \) and \( x \). Recall that the Lie algebra of the maximal group admitted by the system (1.2) is termed the principal Lie algebra for (1.2) and is denoted \( L_\mathcal{P} \). Thus, the algebra \( L_\mathcal{P} \) contains at least two operators:

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}.
\]

1.2 Equivalence generator

An equivalence transformation of the equations (1.2) is a change of variables \((t, x, u, c) \to (\tilde{t}, \tilde{x}, \tilde{u}, \tilde{c})\) taking the system (1.2) into a system of the same form, generally speaking, with different functions \( \tilde{f}(\tilde{u}) \) and \( \tilde{g}(\tilde{c}, \tilde{u}) \). The generators of the continuous group of equivalence transformations have the form

\[
Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial c} + \mu^1 \frac{\partial}{\partial f} + \mu^2 \frac{\partial}{\partial g},
\]

where

\[
\xi^i = \xi^i(t, x, u, c), \quad \eta^i = \eta^i(t, x, u, c), \quad \mu^i = \mu^i(t, x, u, c, f, g), \quad i = 1, 2.
\]

The conditions (see [5]) for the operator (1.4) to generate the equivalence group is that it is admitted by the extended system:

\[
\begin{align*}
&u_t - f + u_x c_x + uc_{xx} = 0, \quad c_t + g = 0 \\
&f_t = 0, \quad f_x = 0, \quad f_c = 0, \quad g_t = 0, \quad g_x = 0
\end{align*}
\]

(1.5)

The infinitesimal invariance test for the system (1.5) requires the following prolongation of the operator (1.4):

\[
\tilde{Y} = Y + \zeta^1 \frac{\partial}{\partial u_t} + \zeta^2 \frac{\partial}{\partial u_x} + \zeta^1 \frac{\partial}{\partial c_t} + \zeta^2 \frac{\partial}{\partial c_x} + \zeta^3 \frac{\partial}{\partial f} + \zeta^4 \frac{\partial}{\partial g} + \xi^1 \frac{\partial}{\partial f_t} + \omega^1 \frac{\partial}{\partial f_x} + \omega^2 \frac{\partial}{\partial f_u} + \omega^3 \frac{\partial}{\partial f_c} + \xi^1 \frac{\partial}{\partial g_t} + \omega^1 \frac{\partial}{\partial g_x} + \omega^2 \frac{\partial}{\partial g_u} + \omega^3 \frac{\partial}{\partial g_c}.
\]

(1.6)

Using the notation

\[
x^1 = t, \quad x^2 = x, \quad u^1 = u, \quad u^2 = c, \quad f^1 = f, \quad f^2 = g,
\]
and 
\[(f^1, f^2) = (f, g), \quad (\nu^1, \nu^2, \nu^3, \nu^4) = (\xi^1, \xi^2, \eta^1, \eta^2),\]
the additional coordinates \(\zeta\) and \(\omega\) can be expressed by the prolongation formulae:
\[
\zeta^k_i = D_i(\eta^k) - u^k_j D_i(\xi^j), \quad i, j, k = 1, 2; \tag{1.7}
\]
\[
\zeta^k_{ij} = D_j(\zeta^k_i) - u^k_{il} D_j(\xi^l), \quad i, j, k, l = 1, 2.
\]

and
\[
\omega^k_\alpha = \tilde{D}_\alpha(\mu^k) - f^k_\beta \tilde{D}_\alpha(\nu^\beta)
\]
\[
\equiv \tilde{D}_\alpha(\mu^k) - f^k_t \tilde{D}_\alpha(\xi^1) - f^k_x \tilde{D}_\alpha(\xi^2) - f^k_u \tilde{D}_\alpha(\eta^1) - f^k_c \tilde{D}_\alpha(\eta^2), \tag{1.8}
\]
\[
k = 1, 2; \quad \alpha = 1, 2, 3, 4.
\]
Here
\[
D_i = \frac{\partial}{\partial x^i} + u^k_i \frac{\partial}{\partial u^k} + u^k_{ij} \frac{\partial}{\partial u^j}
\]
\[
(1.9)
\]
is the usual total differentiation whereas
\[
\tilde{D}_t = \frac{\partial}{\partial t} + f_t \frac{\partial}{\partial f} + g_t \frac{\partial}{\partial g}, \quad \tilde{D}_x = \frac{\partial}{\partial x} + f_x \frac{\partial}{\partial f} + g_x \frac{\partial}{\partial g},
\]
\[
\tilde{D}_u = \frac{\partial}{\partial u} + f_u \frac{\partial}{\partial f} + g_u \frac{\partial}{\partial g}, \quad \tilde{D}_c = \frac{\partial}{\partial c} + f_c \frac{\partial}{\partial f} + g_c \frac{\partial}{\partial g}.
\]
denote the “new” total differentiations for the extended system (1.5). In view of equations (1.5) the latter differentiations are written:
\[
\tilde{D}_t = \frac{\partial}{\partial t}, \quad \tilde{D}_x = \frac{\partial}{\partial x}, \quad \tilde{D}_u = \frac{\partial}{\partial u} + f_u \frac{\partial}{\partial f} + g_u \frac{\partial}{\partial g}, \quad \tilde{D}_c = \frac{\partial}{\partial c} + f_c \frac{\partial}{\partial f} + g_c \frac{\partial}{\partial g}. \tag{1.10}
\]

The invariance test for the operator (1.4) to be an equivalence generator yields the following system of determining equations:
\[
\zeta^1_1 - \mu^1 + c_x \zeta^1_2 + u_x \zeta^2_2 + \eta^1 c_{xx} + u \zeta^2_{22} = 0, \quad \zeta^2_1 + \mu^2 = 0, \tag{1.11}
\]
\[
\omega^1_1 = 0, \quad \omega^1_2 = 0, \quad \omega^2_1 = 0, \quad \omega^2_2 = 0. \tag{1.12}
\]

Let us first solve the equations (1.12). We have
\[
\omega^1_t = \tilde{D}_t(\mu^1) - f_u \tilde{D}_t(\eta^1) = \mu^1_t - f_u \eta^1_t = 0,
\]
whence, invoking that \(f(u)\) and hence \(f_u = f'(u)\) are arbitrary functions,
\[
\mu^1_t = 0, \quad \eta^1_t = 0.
\]
Likewise one obtains:
\[ \omega_2^1 = \mu_x^1 - f_u \eta_x^1 = 0 \quad \Rightarrow \quad \mu_x^1 = 0, \; \eta_x^1 = 0, \]
\[ \omega_4^1 = \mu_c^1 + g_c \mu_y^1 - f_u \eta_c^1 = 0 \quad \Rightarrow \quad \mu_c^1 = 0, \; \mu_y^1 = 0, \; \eta_c^1 = 0, \]
\[ \omega_1^2 = \mu_t^2 - g_c \eta_t^2 = 0 \quad \Rightarrow \quad \mu_t^2 = 0, \; \eta_t^2 = 0, \]
\[ \omega_2^2 = \mu_x^2 - g_c \eta_x^2 = 0 \quad \Rightarrow \quad \mu_x^2 = 0, \; \eta_x^2 = 0. \]

Hence,
\[ \mu^1 = \mu^1(u, f), \quad \mu^2 = \mu^2(u, c, f, g), \quad \eta^1 = \eta^1(u), \quad \eta^2 = \eta^2(u, c). \]

The prolongation formulae (1.7) give
\[ \zeta_1^1 = D_t(\eta^1) - u_tD_t(\xi^1) - u_xD_x(\xi^2), \]
\[ \zeta_2^1 = D_x(\eta^1) - u_tD_x(\xi^1) - u_xD_x(\xi^2), \]
\[ \zeta_1^2 = D_t(\eta^2) - c_tD_t(\xi^1) - c_xD_x(\xi^2), \]
\[ \zeta_2^2 = D_x(\eta^2) - c_tD_x(\xi^1) - c_xD_x(\xi^2), \] (1.13)

and
\[ \zeta_{22}^2 = u_{xx} \eta_u^2 + c_{xx} \eta_c^2 + (u_x)^2 \eta_{uu}^2 + 2u_x c_x \eta_{uc}^2 + (c_x)^2 \eta_{cc}^2 \]
\[ -2c_{tx} D_x(\xi^1) - 2c_{xx} D_x(\xi^2) - c_tD_x^2(\xi^1) - c_x D_x^2(\xi^2), \] (1.14)
respectively.

It follows that the first equation in (1.11), invoking (1.13), can be written
\[ D_t(\eta^1) - u_tD_t(\xi^1) - u_xD_x(\xi^2) + c_x \left[ D_x(\eta^1) - u_tD_x(\xi^1) - u_xD_x(\xi^2) \right] \]
\[ -\mu^1(u, f) + u_x \left[ u_{xx} \eta_u^1 + c_{xx} \eta_c^1 - c_tD_x(\xi^1) - c_x D_x(\xi^2) \right] + \eta^1 c_{xx} + u_{xx}^2 = 0 \] (1.15)

Single out in (1.15) the terms with second derivatives of \( u \) and \( c \). These are terms with \( u_{xx}, c_{tx} \) and \( c_{xx} \). The terms with \( u_{xx} \) and \( c_{tx} \) appear only in \( \zeta_{22}^2 \) (1.14). Equating to zero this terms with \( u_{xx} \), we get (see (1.14))
\[ \eta_u^2 = 0, \quad \xi_u^1 = 0, \quad \xi_u^2 = 0. \]

Likewise we get, using in (1.14) the terms with \( c_{tx} \):
\[ D_x(\xi^1) = \xi_x^1 + u_x \xi_x^1 + c_x \xi_x^1 = 0 \]
whence \( \xi_x^1 = \xi_u^1 = \xi_c^1 = 0 \). Thus:
\[ \xi^1 = \xi^1(t), \quad \xi^2 = \xi^2(t, x, c), \quad \eta^2 = \eta^2(c). \] (1.16)
Now we single out the terms with $c_{xx}$ to obtain:

$$-u \left( \eta_u^1 - \xi_t^1 - u_x \xi_u^2 \right) + \eta^1(u) + u \eta_c^2 - 2u \left( \xi_x^2 + u_x \xi_u^2 + c_x \xi_c^2 \right) - uu_x c_x \xi_c^2 = 0. \quad (1.17)$$

Separating terms with $u_x$ and $c_x$ from the latter equation gives

$$\xi_u^2 = 0, \quad \xi_c^2 = 0$$

and it follows

$$\xi^1 = \xi^1(t), \quad \xi^2 = \xi^2(t, x), \quad \eta^1 = \eta^1(u),$$

$$\eta^2 = \eta^2(c), \quad \mu^1 = \mu^1(u, f), \quad \mu^2 = \mu^2(u, c, f, g). \quad (1.18)$$

The equation (1.17) is rewritten

$$\eta^1(u) + u \left( \eta_c^2 + \xi_t^1 - \eta_u^1 - 2\xi_x^2 \right) = 0 \quad (1.19)$$

Differentiating the above equation (1.19) with respect to $x$, one obtain

$$\xi^2_{xx} = 0,$$

whence $\xi^2(t, x) = a_1(t)x + a_2(t)$, where $a_1$ and $a_2$ are arbitrary functions. Further, differentiate (1.19) with respect to $c$ to acquire the conditions for $\eta^2$:

$$\eta^2_{cc} = 0 \quad \text{or} \quad \eta^2(c) = K_1 c + K_2,$$

where $K_1, K_2$ are arbitrary constants.

Substituting the expressions for $\xi^2$ and $\eta^2$ in to (1.19) we get:

$$\eta^1 + u \left[ K_1 + \xi_t^1 - \eta_u^1 - 2a_1(t) \right] = 0, \quad (1.20)$$

and it follows from (1.20) that ($\eta^1$ does not depend on $t$)

$$\xi_t^1 = 2a_1(t) + K_3,$$

where $K_3$ is an arbitrary constant.

The equation (1.20) is rewritten:

$$u \frac{d\eta^1}{du} = \eta^1 + (K_1 + K_3) u$$

and

$$\eta^1 = K_4 u + (K_1 + K_3) u \ln u$$
Hence
\[ \xi^1 = \xi^1(t), \quad \xi^2 = \frac{1}{2} \left[ \dot{\xi}^1(t) - K_3 \right] x + a_2(t), \]
\[ \eta^1 = K_4 u + (K_1 + K_3) u \ln u, \quad \eta^2 = K_1 c + K_2, \]
\[ \mu^1 = \mu^1(u, f), \quad \mu^2 = \mu^2(u, c, f, g), \]
where \( \dot{\xi}^1(t) = d\xi^1/dt \).

Substituting all above expression into (1.15):
\[ \left[ K_1 + K_3 + K_4 + (K_1 + K_3) u \ln u \right] f - \dot{\xi}^1 f - \mu^1(u, f) \]
\[ - \left[ \frac{1}{2} \dot{\xi}^1 x + a'_2(t) \right] u_x + \left[ K_1 + K_3 \right] u_x c_x = 0. \]

Collecting terms with \( x, \dot{\xi}^1 = 0 \) hence
\[ \xi^1 = K_5 t + K_6. \]

Terms with \( u_x c_x \) and \( u_x \) gives
\[ K_1 + K_3 = 0 \quad \text{and} \quad a'_2(t) = 0, \]
respectively, and it follows that \( a_2 = B = \text{constant} \). From the equation (1.23) we finally get
\[ -\mu^1 + K_4 f - \dot{\xi}^1 f = 0. \]

Thus
\[ \xi^1 = K_5 t + K_6, \quad \xi^2 = \frac{1}{2} (K_5 - K_3) x + B, \]
\[ \eta^1 = K_4 u, \quad \eta^2 = K_2 - K_3 c, \]
\[ \mu^1 = (K_4 - K_5) f. \]

The second determining equation (1.11), invoking (1.24), gives the condition for \( \mu^2 \):
\[ \mu^2 = K_5 c_t - K_3 c_t = - (K_3 + K_5) g. \]

Finally, the equations (1.24) and (1.25) yield the following infinitesimal generators of the equivalence group for the system (1.2):
\[ Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2f \frac{\partial}{\partial f} - 2g \frac{\partial}{\partial g}, \]
\[ Y_4 = \frac{\partial}{\partial c}, \quad Y_5 = x \frac{\partial}{\partial x} + 2c \frac{\partial}{\partial c} + 2g \frac{\partial}{\partial g}, \quad Y_6 = u \frac{\partial}{\partial u} + f \frac{\partial}{\partial f}. \]

In Table 1, the commutator is defined by \([Y_\alpha, Y_\beta] = Y_\alpha Y_\beta - Y_\beta Y_\alpha\).
1.3 Application

We will use the theorem on projections of equivalence generators. Let us denote \( x = (t, x) \) and \( u = (u, c) \) the independent and dependent variables in the system (1.2). Consider the projection \( X = \text{pr}_{(x, u)}(Y) \) of the equivalence generator (1.4) to the space \((x, u)\) of the independent variables, and the projection \( Z = \text{pr}_{(u, f)}(Y) \) to the space \((u, f)\) involved in the arbitrary elements:

\[
X = \text{pr}_{(x, u)}(Y) \equiv \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial c}, \tag{1.27}
\]

\[
Z = \text{pr}_{(u, f)}(Y) \equiv \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial c} \mu^1 \frac{\partial}{\partial f} + \mu^2 \frac{\partial}{\partial g}, \tag{1.28}
\]

The equations (1.24) and (1.25) manifest that the operators \( X \) and \( Z \) are well defined, i.e. their coordinates involve only the respective variables, \((x, u)\) and \((u, f)\). In our specific case the theorem on projections is formulated as follows.

Given any equivalence generator

\[
Y = K_1 Y_1 + K_2 Y_2 + K_3 Y_3 + K_4 Y_4 + K_5 Y_5 + K_6 Y_6
\]

(a linear combination of the operators (1.26)):

\[
Y = (K_1 + 2K_3t) \frac{\partial}{\partial t} + [K_2 + (K_3 + K_5)x] \frac{\partial}{\partial x} + K_6 u \frac{\partial}{\partial u} + (K_4 + 2K_5 c) \frac{\partial}{\partial c}
+ (K_6 - 2K_3) f \frac{\partial}{\partial f} + 2(K_5 - K_3) g \frac{\partial}{\partial g}, \tag{1.29}
\]

its projection (1.27)

\[
X = (K_1 + 2K_3t) \frac{\partial}{\partial t} + [K_2 + (K_3 + K_5)x] \frac{\partial}{\partial x} + K_6 u \frac{\partial}{\partial u} + (K_4 + 2K_5 c) \frac{\partial}{\partial c} \tag{1.30}
\]

The general theorem on projections of equivalence Lie algebras was first demonstrated by N.H. Ibragimov in 1987 (unpublished) and applied to several particular problems, e.g. for preliminary group classification of families of nonlinear differential equations [7, 8]. In this problem we use the theorem on projections in a way similar to that in [9].
is admitted by the system (1.2) with specific functions

\[ f = F(u), \quad g = G(c, u) \quad (1.31) \]

if and only if the constants \( K \) are chosen so that the projection (1.28),

\[ Z = +K_6 u \frac{\partial}{\partial u} + (K_4 + 2K_5 c) \frac{\partial}{\partial c} + (K_6 - 2K_3) f \frac{\partial}{\partial f} + 2(K_5 - K_3) g \frac{\partial}{\partial g} \quad (1.32) \]

is admitted by the equations (1.31). In particular, \( X \in L_\varphi \) if and only if the projections (1.32) vanish:

\[ Z = \text{pr}(u, f)(Y) = 0. \quad (1.33) \]

### 1.3.1 The principal Lie algebra \( L_\varphi \)

Equation (1.33) provides a simple way to find the principal Lie algebra. Indeed, according to (1.32), equation (1.33) is written

\[ K_6 u \frac{\partial}{\partial u} + (K_4 + 2K_5 c) \frac{\partial}{\partial c} + (K_6 - 2K_3) f \frac{\partial}{\partial f} + 2(K_5 - K_3) g \frac{\partial}{\partial g} = 0, \]

whence

\[ K_3 = 0, \quad K_4 = 0, \quad K_5 = 0, \quad K_6 = 0, \]

and hence \( Y = K_1 Y_1 + K_2 Y_2 \). Since \( X_1 = \text{pr}(x, u)(Y_1) = Y_1 \) and \( X_2 = \text{pr}(x, u)(Y_2) = Y_2 \), we conclude that the principal Lie algebra \( L_\varphi \) is spanned by the operator (1.3):

\[ X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}. \]

### 1.3.2 An invariant solution

Consider, as an example, the operator

\[ Y = Y_4 + Y_6 \equiv \frac{\partial}{\partial c} + u \frac{\partial}{\partial u} + f \frac{\partial}{\partial f}. \quad (1.34) \]

In this example, the operator \( Y \) coincides with its projection \( \text{pr}(u, f)(Y) \):

\[ Z = \text{pr}(u, f)(Y) = \frac{\partial}{\partial c} + u \frac{\partial}{\partial u} + f \frac{\partial}{\partial f}. \]

The invariance conditions for equations (1.31) are written

\[ Z \left( f - F(u) \right) \bigg|_{f=F(u)} = F - uF'(u) = 0 \]
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\[ Z (g - G(c, u)) \bigg|_{g=G(c,u)} = -\frac{\partial G}{\partial c} - u \frac{\partial G}{\partial u} = 0, \]

whence

\[ f = \alpha u, \quad g = G(ue^{-c}), \quad (1.35) \]

where \( \alpha \) is an arbitrary constant and \( G \) an arbitrary function of one variable. The corresponding system (1.2),

\[ u_t = \alpha u - (uc_x)_x, \]
\[ c_t = -G(ue^{-c}) \]

admits, along with the operators \( X_1, X_2 \) given in (1.3), the additional operator

\[ X_3 = \text{pr}(x;u) (Y) = \frac{\partial}{\partial c} + u \frac{\partial}{\partial u}. \]

Letting, e.g. \( G(ue^{-c}) = ue^{-c} \) we consider the system

\[ u_t = \alpha u - (uc_x)_x, \]
\[ c_t = -ue^{-c} \quad (1.36) \]

Let us find invariant solutions using the operator

\[ X_1 + X_3 = \frac{\partial}{\partial t} + \frac{\partial}{\partial c} + u \frac{\partial}{\partial u}. \]

It has three independent invariants:

\[ x, \quad \psi_1 = c - t, \quad \psi_2 = ue^{-t}. \]

The corresponding invariant solutions have the form

\[ c = t + \psi_1(x), \quad u = e^{t} \psi_2(x). \quad (1.37) \]

It follows

\[ u_t = e^{t} \psi_2(x), \quad u_x = e^{t} \psi'_2(x), \quad c_t = 1, \quad c_x = \psi'_1(x), \quad c_{xx} = \psi''_1(x). \quad (1.38) \]

Substituting the expressions (1.37) and (1.38) in the first equation (1.36), one obtains:

\[ e^{t} \psi_2(x) = \alpha e^{t} \psi_2(x) - e^{t} \psi'_1(x) \psi'_2(x) - e^{t} \psi_2(x) \psi''_1(x), \]

or

\[ (1 - \alpha) \psi_2(x) + \psi'_1(x) \psi'_2(x) + \psi_2(x) \psi''_1(x). \quad (1.39) \]
The second equation (1.36) implies

\[ 1 = -\psi_2(x)e^{-\psi_1(x)} \]

whence

\[ \psi_2(x) = -e^{\psi_1(x)}. \] (1.40)

The equation (1.39), invoking (1.40), is rewritten

\[ \psi''_1(x) + \psi'_1 + (1 - \alpha) = 0 \]

and yields:

1. for \( \alpha = 1 \):
   \[ \psi_1(x) = \ln |A_2(x + A_1)|, \] (1.41)

2. for \( \alpha > 1 \):
   \[ \psi_1(x) = x\sqrt{\alpha - 1} + \ln \left| A_2 \left( 1 \pm e^{2(A_1-x)\sqrt{\alpha-1}} \right) \right|, \] (1.42)

3. for \( \alpha < 1 \):
   \[ \psi_1(x) = \ln \left| A_2 \cos \left( (A_1 - x)\sqrt{1-\alpha} \right) \right|, \] (1.43)

where \( A_1 \) and \( A_2 \) are arbitrary constants.

Finally, substituting (1.41), (1.42) and (1.43) into the equations for \( c \) and \( u \) (1.37), invoking (1.40), one obtains three different solutions for the system (1.36):

\[ c(t,x) = t + \ln |A_2(x + A_1)|, \]
\[ u(t,x) = -|A_2(x + A_1)|e^t \quad (\alpha = 1), \] (1.44)

\[ c(t,x) = t + x\sqrt{\alpha - 1} + \ln \left| A_2 \left( 1 \pm e^{2(A_1-x)\sqrt{\alpha-1}} \right) \right|, \]
\[ u(t,x) = -\left| A_2 \left( 1 \pm e^{2(A_1-x)\sqrt{\alpha-1}} \right) \right|e^{t+x\sqrt{\alpha-1}} \quad (\alpha > 1) \] (1.45)

and

\[ c(t,x) = t + \ln \left| A_2 \cos \left( (A_1 - x)\sqrt{1-\alpha} \right) \right|, \]
\[ u(t,x) = -e^t \ln \left| A_2 \cos \left( (A_1 - x)\sqrt{1-\alpha} \right) \right| \quad (\alpha < 1), \] (1.46)

respectively. The solution (1.46) with \( \alpha < 0 \) is relevant for the model (1.36). Namely, the functions \( f(u) = \alpha u \) and \( g = ue^{-c} \) satisfy the condition for the model (1.2):

\[ f(u) = \alpha u > 0, \quad g_c(c,u) = -ue^{-c} > 0, \quad g_u(c,u) = e^{-c} > 0. \]

The function \( c(t,x) \) and \( u(t,x) \) given by (1.46) are illustrated in Fig. 0.1 and 0.2 (where \( A_1 = A_2 = 1 \) and \( \sqrt{1-\alpha} = 2 \)).
2 General model

2.1 Mathematical model

Let us turn now to the following more general model formulated in [1], Section 2:

\[ \frac{\partial u}{\partial t} = f(u) - k \frac{\partial}{\partial x} \left( u \frac{\partial c}{\partial x} \right), \]

\[ \frac{\partial c}{\partial t} = -g(c, p), \]

\[ \frac{\partial p}{\partial t} = h(u, c) - K p, \]

where \( k \) and \( K \) are positive constants.

Ignoring diffusion, Perumpanini et al. [1] described the malignant invasion due to processes of haptotaxis and proteolysis. Here, the functions \( f(u), g(c, p) \) and \( h(u, c) \) corresponds to invasive cell proliferation, proteolysis and protease production, respectively. The model (2.1) was later investigated by Stewart et al. [3] for specific functions \( f, g, h, \) including a transient protease concentration.

The last equation of the above system (2.1) is written under the assumption that protease decays linearly, with half-life \( K \). We will discard the latter assumption and consider an arbitrary law of the protease decay. Furthermore, we exclude the immaterial constant \( k \), e.g. by stretching \( x \). So, we will discuss further the following model for arbitrary functions \( f(u), g(c, p), h(u, c) \) and \( n(p) \):
\[ \frac{\partial u}{\partial t} = f(u) - \frac{\partial}{\partial x} \left( u \frac{\partial c}{\partial x} \right), \]
\[ \frac{\partial c}{\partial t} = -g(c, p), \]
\[ \frac{\partial p}{\partial t} = h(u, c) - n(p). \]

2.2 Equivalence generator

We consider a continuous local group of equivalence transformations and seek for its generator of the form
\[ Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial c} + \eta^3 \frac{\partial}{\partial p} + \mu^1 \frac{\partial}{\partial f} + \mu^2 \frac{\partial}{\partial g} + \mu^3 \frac{\partial}{\partial h} + \mu^4 \frac{\partial}{\partial n}, \] (2.3)
where
\[ \xi = \xi(t, x, u, c, p), \quad \eta = \eta(t, x, u, c, p), \quad \mu = \mu(t, x, u, c, p, f, g, h, n) \]

The operator (2.3) generates an equivalence group, if and only if it is admitted by the following extended system:
\[ u_t - f + u_x c_x + u c_{xx} = 0, \quad c_t + g = 0, \quad p_t - h + n = 0, \]
\[ f_t = 0, \quad f_x = 0, \quad f_c = 0, \quad f_p = 0, \quad g_t = 0, \quad g_x = 0, \quad g_u = 0, \]
\[ h_t = 0, \quad h_x = 0, \quad h_p = 0, \quad n_t = 0, \quad n_x = 0, \quad n_u = 0, \quad n_c = 0. \] (2.4)

To perform the invariance test for the system (2.4), we have to prolong the operator (2.3) up to the second order:
\[ \tilde{Y} = Y + \zeta^1 \frac{\partial}{\partial u_t} + \zeta^2 \frac{\partial}{\partial u_x} + \zeta^3 \frac{\partial}{\partial c_t} + \zeta^4 \frac{\partial}{\partial c_x} + \zeta^5 \frac{\partial}{\partial c_{xx}} + \zeta^6 \frac{\partial}{\partial c_{xxx}} + \zeta^7 \frac{\partial}{\partial c_{xxxx}} + \zeta^8 \frac{\partial}{\partial f_t} + \zeta^9 \frac{\partial}{\partial f_x} + \zeta^{10} \frac{\partial}{\partial f_c} + \zeta^{11} \frac{\partial}{\partial f_p} + \zeta^{12} \frac{\partial}{\partial g_t} + \zeta^{13} \frac{\partial}{\partial g_x} + \zeta^{14} \frac{\partial}{\partial g_c} + \zeta^{15} \frac{\partial}{\partial g_p} + \zeta^{16} \frac{\partial}{\partial h_t} + \zeta^{17} \frac{\partial}{\partial h_x} + \zeta^{18} \frac{\partial}{\partial h_c} + \zeta^{19} \frac{\partial}{\partial h_p} + \zeta^{20} \frac{\partial}{\partial n_t} + \zeta^{21} \frac{\partial}{\partial n_x} + \zeta^{22} \frac{\partial}{\partial n_c} + \zeta^{23} \frac{\partial}{\partial n_p}. \] (2.5)
Using the notation

\[(x^1, x^2, u^1, u^2, u^3) = (t, x, u, c, p), \quad (f^1, f^2, f^3, f^4) = (f, g, h, n), \]

\[(\nu^1, \nu^2, \nu^3, \nu^4, \nu^5) = (\xi^1, \xi^2, \eta^1, \eta^2, \eta^3)\]

the prolongation formulae, for the additional coordinates \(\zeta\) and \(\omega\), are written:

\[\zeta^k_i = D_i(\eta^k) - u^k_i D_i(\xi^j), \quad i, j = 1, 2; \quad k = 1, 2, 3; \quad (2.6)\]

and

\[\omega^k_\alpha = \tilde{D}_\alpha(\mu^k) - f^k_\beta \tilde{D}_\beta(\nu^\beta)\]

\[\equiv \tilde{D}_\alpha(\mu^k) - f^k_\beta \tilde{D}_\beta(\xi^l) - f^k_\alpha \tilde{D}_\alpha(\xi^l) - f^k_\beta \tilde{D}_\alpha(\eta^\beta) - f^k_\beta \tilde{D}_\alpha(\eta^\beta), \quad (2.7)\]

\[k = 1, 2, 3, 4; \quad \alpha = 1, 2, 3, 4, 5.\]

Here

\[\tilde{D}_t^{(2.4)} = \frac{\partial}{\partial t}, \quad \tilde{D}_x^{(2.4)} = \frac{\partial}{\partial x}, \quad \tilde{D}_u^{(2.4)} = \frac{\partial}{\partial u} + f_u \frac{\partial}{\partial f} + h_u \frac{\partial}{\partial h} \]

\[\tilde{D}_c^{(2.4)} = \frac{\partial}{\partial c} + g_c \frac{\partial}{\partial g} + h_c \frac{\partial}{\partial h}, \quad \tilde{D}_p^{(2.4)} = \frac{\partial}{\partial p} + g_p \frac{\partial}{\partial g} + n_p \frac{\partial}{\partial n} \quad (2.8)\]

denote the new total differentiation for the extended system (2.4).

Now, the condition for the operator (2.3) to be an equivalence generator give rise to the following system of determining equations:

\[\zeta^1_1 - \mu^1 + c_x \zeta^2_1 + u_x \zeta^2_2 + c_{xx} \eta^1 + u \zeta^2_2 = 0, \]
\[\zeta^2_1 + \mu^2 = 0, \quad \zeta^3_1 - \mu^3 + \mu^4 = 0 \quad (2.9)\]
\[\omega^1_1 = 0, \quad \omega^2_1 = 0, \quad \omega^3_1 = 0, \quad \omega^4_1 = 0, \quad \omega^5_1 = 0, \quad \omega^2_2 = 0, \quad \omega^3_2 = 0, \quad \omega^4_2 = 0, \quad \omega^5_2 = 0, \quad (2.10)\]

To start with the equations (2.10), we have

\[\omega^1_1 \equiv \tilde{D}_t(\mu^1) - f_u \tilde{D}_t(\eta^1) = \mu^1 - f_u \eta^1_1 = 0\]

and it follows

\[\mu^1_1 = 0, \quad \eta^1_1 = 0 \quad (2.11)\]

Likewise we obtains:
\[ \omega_2 \equiv \tilde{D}_x(\mu^1) - f_u \tilde{D}_x(\eta^1) = \mu^1_x + f_u \eta^1_x = 0 \]
\[ \implies \mu^1_x = 0, \quad \eta^1_x = 0, \]
\[ \omega_4 \equiv \tilde{D}_c(\mu^1) - f_u \tilde{D}_c(\eta^1) = \mu^1_c + g_c \mu^1_g + h_c \mu^1_h - f_u \eta^1_c = 0 \]
\[ \implies \mu^1_c = 0, \quad \mu^1_g = 0, \quad \mu^1_h = 0, \quad \eta^1_c = 0, \]
\[ \omega_5 \equiv \tilde{D}_p(\mu^1) - f_u \tilde{D}_p(\eta^1) = \mu^1_p + g_p \mu^1_g + n_p \mu^1_n + -f_u \eta^1_p = 0 \]
\[ \implies \mu^1_p = 0, \quad \mu^1_g = 0, \quad \mu^1_n = 0, \quad \eta^1_p = 0, \]
\[ \omega_1^2 \equiv \bar{D}_t(\mu^2) - g_c \bar{D}_t(\eta^2) - g_p \bar{D}_t(\eta^3) = \mu_t^2 - g_c \eta_t^2 - g_p \eta_t^3 = 0 \]
\[ \Rightarrow \mu_t^2 = 0, \quad \eta_t^2 = 0, \quad \eta_t^3 = 0, \]

\[ \omega_2^2 \equiv \bar{D}_x(\mu^2) - g_c \bar{D}_x(\eta^2) - g_p \bar{D}_x(\eta^3) = \mu_x^2 - g_c \eta_x^2 - g_p \eta_x^3 = 0 \]
\[ \Rightarrow \mu_x^2 = 0, \quad \eta_x^2 = 0, \quad \eta_x^3 = 0, \]

\[ \omega_3^2 \equiv \bar{D}_u(\mu^2) - g_c \bar{D}_u(\eta^2) - g_p \bar{D}_u(\eta^3) = \mu_u^2 + f_u \mu_u^2 + h_u \mu_u^2 - g_c \eta_u^2 - g_p \eta_u^3 = 0 \]
\[ \Rightarrow \mu_u^2 = 0, \quad \mu_u^2 = 0, \quad \eta_u^2 = 0, \quad \eta_u^3 = 0, \]

\[ \omega_1^3 \equiv \bar{D}_t(\mu^3) - h_u \bar{D}_t(\eta^1) - h_c \bar{D}_t(\eta^2) = \mu_t^3 = 0 \]
\[ \Rightarrow \mu_t^3 = 0, \]

\[ \omega_2^3 \equiv \bar{D}_x(\mu^3) - h_u \bar{D}_x(\eta^1) - h_c \bar{D}_x(\eta^2) = \mu_x^3 = 0 \]
\[ \Rightarrow \mu_x^3 = 0, \]

\[ \omega_5^3 \equiv \bar{D}_p(\mu^3) - h_u \bar{D}_p(\eta^1) - h_c \bar{D}_p(\eta^2) = \mu_p^3 + g_p \mu_p^2 + n_p \mu_p^3 - h_c \eta_p^2 = 0 \]
\[ \Rightarrow \mu_p^3 = 0, \quad \mu_p^2 = 0, \quad \mu_p^3 = 0, \quad \eta_p^2 = 0, \]

\[ \omega_1^4 \equiv \bar{D}_t(\mu^4) - n_p \bar{D}_t(\eta^3) = \mu_t^4 = 0 \]
\[ \Rightarrow \mu_t^4 = 0, \]

\[ \omega_2^4 \equiv \bar{D}_x(\mu^4) - n_p \bar{D}_x(\eta^3) = \mu_x^4 = 0 \]
\[ \Rightarrow \mu_x^4 = 0, \]

\[ \omega_3^4 \equiv \bar{D}_u(\mu^4) - n_p \bar{D}_u(\eta^3) = \mu_u^4 + f_u \mu_u^4 + h_u \mu_u^4 = 0 \]
\[ \Rightarrow \mu_u^4 = 0, \quad \mu_u^4 = 0, \quad \mu_u^4 = 0, \]

\[ \omega_4^4 \equiv \bar{D}_c(\mu^4) - n_p \bar{D}_c(\eta^3) = \mu_c^4 + g_c \mu_c^4 + h_c \mu_c^4 - n_p \eta_c^2 = 0 \]
\[ \Rightarrow \mu_c^4 = 0, \quad \mu_c^4 = 0, \quad \mu_c^4 = 0, \quad \eta_c^2 = 0. \]

Hence,

\[ \mu^1 = \mu^1(u, f), \quad \mu^2 = \mu^2(c, p, g, n), \quad \mu^3 = \mu^3(u, c, f, h), \quad \mu^4 = \mu^4(p, n), \]
\[ \eta^1 = \eta^1(u), \quad \eta^2 = \eta^2(c), \quad \eta^3 = \eta^3(p). \]

(2.12)
The prolongation formulae (2.6) give
\[
\begin{align*}
\zeta_1^1 &= D_t(\eta^1) - u_t D_t(\xi^1) - u_x D_t(\xi^2) = u_t \left[ \eta^1_u - D_t(\xi^1) \right] - u_x D_t(\xi^2) \\
\zeta_1^2 &= D_x(\eta^1) - u_t D_x(\xi^1) - u_x D_x(\xi^2) = u_x \left[ \eta^1_x - D_x(\xi^1) \right] - u_t D_x(\xi^1) \\
\zeta_2^1 &= D_t(\eta^2) - c_t D_t(\xi^1) - c_x D_t(\xi^2) = c_t \left[ \eta^2_t - D_t(\xi^1) \right] - c_x D_t(\xi^2) \\
\zeta_2^2 &= D_x(\eta^2) - c_t D_x(\xi^1) - c_x D_x(\xi^2) = c_x \left[ \eta^2_x - D_x(\xi^1) \right] - c_t D_x(\xi^1) \\
\zeta_3^1 &= D_t(\eta^3) - p_t D_t(\xi^1) - p_x D_t(\xi^2) = p_t \left[ \eta^3_t - D_t(\xi^1) \right] - p_x D_t(\xi^2)
\end{align*}
\tag{2.13}
\]

and
\[
\begin{align*}
\zeta_2^{22} &= c_{xx} \eta^2_c + (c_x)^2 \eta^2_{cc} - c_t D_x(\xi^1) - c_x D_x^2(\xi^2) - 2c_{tx} D_x(\xi^1) - 2c_{xx} D_x(\xi^2),
\end{align*}
\tag{2.14}
\]
respectively.

Substituting the expressions (2.13) and (2.14) into the first equation (2.9) yield:
\[
\begin{align*}
&u_t \eta_u^1 - u_t D_t(\xi^1) - u_x D_t(\xi^2) - \mu^1(u, f) + c_x \left[ u_x \eta^1_u - u_t D_x(\xi^1) - u_x D_x(\xi^2) \right] \\
&+ u_x \left[ c_x \eta^2_x - c_t D_x(\xi^1) - c_x D_x(\xi^2) \right] + c_{xx} \eta^1 \\
&+ u \left[ c_{xx} \eta^2_c + (c_x)^2 \eta^2_{cc} - c_t D_x^2(\xi^1) - c_x D_x^2(\xi^2) - 2c_{tx} D_x(\xi^1) - 2c_{xx} D_x(\xi^2) \right] = 0,
\end{align*}
\tag{2.15}
\]
where
\[
D_t(\xi^j) = \xi^j_t + u_t \xi^j_u + c_t \xi^j_c + p_t \xi^j_p
\]
and
\[
D_x^2(\xi^j) = \xi^j_{xx} + 2u_x \xi^j_x + 2c_x \xi^j_{cx} + 2p_x \xi^j_{px} + u_{xx} \xi^j_x + u_x \xi^j_{sx} + (u_x)^2 \xi^j_{uu} + 2c_x u_x \xi^j_{su} \\
+ 2u_x p_x \xi^j_{sp} + c_{xx} \xi^j_{cc} + (c_x)^2 \xi^j_{cc} + 2c_x p_x \xi^j_{cp} + p_{xx} \xi^j_p + (p_x)^2 \xi^j_{pp}.
\]
Singling out terms with \(u_{xx}\) in (2.15) and equating to zero this terms, we get:
\[-u \left[ c_t \xi^1_u + c_x \xi^2_u \right] = 0,
\]
whence
\[
\xi^1_{su} = 0, \quad \xi^2_{su} = 0.
\]
Likewise we get, using terms with \(c_{tx}\) in (2.15),
\[
D_x(\xi^1) = \xi^1_x + c_x \xi^1_c + p_x \xi^1_p = 0.
\]
Thus:
\[
\xi^1 = \xi^1(t), \quad \xi^2(t, x, c, p)
\tag{2.16}
\]
Continuing with terms involving \(c_{xx}\) to obtain:

\[
\eta^1 + u \left[ \xi^1_t - \eta^1_x + \eta^2_c^2 - c_x \xi^2_c - 2 (\xi^2_x + c_x \xi^2_c + p_x \xi^2_p) \right] = 0. \tag{2.17}
\]

Separating terms with \(c_x\) and \(p_x\) from the latter equation (2.17) gives:

\[
2 \xi^2_c = 0 \quad \text{and} \quad 2 \xi^2_p = 0;
\]

respectively, and it follows

\[
\xi^1 = \xi^1(t), \quad \xi^2 = \xi^2(t, x), \quad \eta^1 = \eta^1(u), \quad \eta^2 = \eta^2(c), \quad \eta^3 = \eta^3(p),
\]

\[
\mu^1 = \mu^1(u, f), \quad \mu^2 = \mu^2(c, p, g, n), \quad \mu^3 = \mu^3(u, c, f, h), \quad \mu^4 = \mu^4(p, n). \tag{2.18}
\]

The equation (2.17) is rewritten

\[
\eta^1(u) + u \left[ \eta^2_c(t) + \xi^1_t(u) - \eta^1_x(u) - 2 \xi^2_x(t, x) \right] = 0.
\]

According to previous calculations for the first system (see Section 1.2, equation (1.19) and further) we have:

\[
\xi^1 = K_5 t + K_6, \quad \xi^2 = \frac{1}{2} (K_5 - K_3) x + B,
\]

\[
\eta^1 = K_4 u, \quad \eta^2 = K_2 - K_3 c,
\]

\[
\mu^1 = (K_4 - K_5) f. \tag{2.19}
\]

The second equation (2.9), invoking (2.19), yield:

\[
\mu^2 = K_5 c_t - K_3 c_t = -(K_3 + K_5) g. \tag{2.20}
\]

Finally, substituting (2.19) and (2.20) into the third equation (2.9):

\[
(h - n) \left[ \eta^3_p(p) - K_5 \right] - \mu^3(u, c, f, h) + \mu^4(p, n) = 0. \tag{2.21}
\]

To determine \(\mu^4\) we differentiate the latter equation twice with respect to \(n\), which implies \(\mu^4_{nn} = 0\). Thus:

\[
\mu^4 = b_1(p)n + b_2(p), \tag{2.22}
\]

where \(b_1\) and \(b_2\) are arbitrary functions of \(p\). Now we differentiate the equation (2.21) with respect to \(p\) to obtain

\[
(h - n) \eta^3_{pp}(p) + b'_1(p)n + b'_2(p) = 0,
\]

whence

\[
\eta^3_{pp}(p) = 0, \quad b'_1(p) = 0, \quad b'_2(p) = 0.
\]
Hence
\[ \eta^3 = L_3 p + L_4, \quad \mu^4 = L_1 n + L_2. \] (2.23)

The equation (2.21) is rewritten
\[ h (L_3 - K_5) - n (L_3 - K_5) - \mu^3 + L_1 n + L_2 = 0, \]
and hence
\[ \mu^3 = (L_3 - K_5) h + L_2, \quad L_1 = L_3 - L_5. \]

We finally have
\[ \xi^1 = K_5 t + K_6, \quad \xi^2 = \frac{1}{2} (K_5 - K_3) x + B, \]
\[ \eta^1 = K_4 u, \quad \eta^2 = K_2 - K_3 c, \quad \eta^3 = L_3 p + L_4, \]
\[ \mu^1 = (K_4 - K_5) f, \quad \mu^2 = -(K_3 + K_5) g, \]
\[ \mu^3 = (L_3 - K_5) h + L_2, \quad \mu^4 = (L_3 - K_5) n + L_2. \] (2.24)

The equations (2.24) yield the following infinitesimal generators of the equivalence group for the system (2.2):
\[ \begin{align*}
Y_1 &= \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = \frac{\partial}{\partial c}, \quad Y_4 = x \frac{\partial}{\partial x} + 2 c \frac{\partial}{\partial c} + 2 g \frac{\partial}{\partial g}, \\
Y_5 &= u \frac{\partial}{\partial u} + f \frac{\partial}{\partial f}, \quad Y_6 = \frac{\partial}{\partial h} + \frac{\partial}{\partial n}, \quad Y_7 = \frac{\partial}{\partial p}, \\
Y_8 &= 2 t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2 f \frac{\partial}{\partial f} - 2 g \frac{\partial}{\partial g} - 2 h \frac{\partial}{\partial h} - 2 n \frac{\partial}{\partial n}, \\
Y_9 &= p \frac{\partial}{\partial p} + h \frac{\partial}{\partial h} + n \frac{\partial}{\partial n}.
\end{align*} \] (2.25)

The commutators of the operators (2.25) are listed in table 2.

### 2.2.1 Principal Lie algebra \( L_\mathcal{P} \)

We search for the principal Lie algebra \( L_\mathcal{P} \), consisting of all operators admitted by the system (2.2), for arbitrary functions \( f(u), g(c, p), h(u, c), n(p) \). Using the notations \( x = (t, x), \ u = (u, c, p), \ f = (f, g, h, n) \), we employ the method from section 1.3 (page 132). It follows that an operator \( X \) belongs to \( L_\mathcal{P} \) if and only if
\[ X = \text{pr}_{(x, u)}(Y), \]
where
\[ Y = \sum_{i=1}^{9} K_i Y_i \quad (K_i = \text{const.}) \]
such that

\[ \text{pr}_{(u,f)}(Y) = 0. \]

(2.26)

The necessary condition for the latter equation (2.26) is that

\[ K_3 = 0, \quad K_4 = 0, \quad K_5 = 0, \quad K_6 = 0, \quad K_7 = 0, \quad K_8 = 0, \quad K_9 = 0 \]

and it follows that the principal Lie algebra \( L_P \) is two dimensional and spanned by

\[ X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}. \]

(2.27)

### 2.3 Optimal system

To widen the group admitted by the system (2.2), one obtain certain condition for functions \( f, g, h, n \). Applying the theorem from section 1.3, it follows (excluding the principal Lie algebra \( L_P \)):

For any equivalence generator

\[ Y = \sum_{\alpha=2}^{9} K_\alpha Y_\alpha \quad (K_\alpha = \text{const.}) \]

its projection

\[ X = \text{pr}_{(x,u)}(Y) \]

will be admitted by the system

\[ \begin{align*}
  u_t &= F(u) - (uc_x)_x, \\
  c_t &= -G(c,p), \\
  p_t &= H(u,c) - N(p)
\end{align*} \]

(2.28)
if and only if the operator
\[ Z = \text{pr}(u, f)(Y) \]
is admitted by the equations
\[ f = F(u), \quad g = G(c, p), \quad h = H(u, c), \quad n = N(p). \]

Now, let \( S \) be any specific system of the type (2.28) and suppose that \( S \) is invariant with respect to the group \( G^S \), i.e.
\[ T_a S = S \quad (T_a \in G^S). \]

If \( T \) is an equivalence transformation, and \( S_T = TS \) the system obtained from \( S \) by means of \( T \), then
\[ TT_a T^{-1} S_T = TT_a S = S_T. \]

Hence, the system \( S_T \) is invariant with respect to the group \( TG^S T^{-1} \). The groups \( G^S \) and \( TG^S T^{-1} \) are said to be similar. Thus, it follows that equivalent system are invariant with respect to similar groups (see [6]).

For any Lie group \( G \), each transformation \( T \in G \) yields an inner automorphism \( T_a \rightarrow TT_a T^{-1} \) of the group \( G \) and every automorphism of the group \( G \) induces an automorphism of its Lie algebra \( L \). The set of all automorphisms is a local Lie group, called the adjoint group, denoted \( G^A \). A basis of the adjoint algebra \( L^A \), connected to \( G^A \), can be represented by the following operator:
\[ A_\alpha = [Z_\alpha, Z_\beta] \frac{\partial}{\partial Z_\beta} \quad (2.29) \]

To classify similar subalgebras (or equivalent systems), it requires to find whenever there is a transformation from \( G^A \) which takes one subalgebra into another. In the following procedure we are searching for the collection of pairwise non-similar one-dimensional subalgebras, so called optimal system of order one (denoted \( \Theta \_1 \)):

Letting
\[ Z_\alpha = \text{pr}(u, f)(Y_\alpha) \quad (Y_\alpha \in L_9 (2.25)) \]
one obtain the following nonzero operators \( Z_\alpha \):
\[ Z_3 = \frac{\partial}{\partial c}, \quad Z_4 = 2c \frac{\partial}{\partial c} + 2g \frac{\partial}{\partial g}, \quad Z_5 = u \frac{\partial}{\partial u} + f \frac{\partial}{\partial f}, \]
\[ Z_6 = \frac{\partial}{\partial h} + \frac{\partial}{\partial n}, \quad Z_7 = \frac{\partial}{\partial p}, \quad Z_8 = -2f \frac{\partial}{\partial f} - 2g \frac{\partial}{\partial g} - 2h \frac{\partial}{\partial h} - 2n \frac{\partial}{\partial n}, \]
\[ Z_9 = p \frac{\partial}{\partial p} + h \frac{\partial}{\partial h} + n \frac{\partial}{\partial n}. \quad (2.30) \]

The commutators of the operators (2.30) are given in table 3.
Table 3: Table of commutators, $L_7. [Z_\alpha, Z_\beta] = Z_\alpha Z_\beta - Z_\beta Z_\alpha$.

<table>
<thead>
<tr>
<th>[r, c]</th>
<th>$Z_3$</th>
<th>$Z_4$</th>
<th>$Z_5$</th>
<th>$Z_6$</th>
<th>$Z_7$</th>
<th>$Z_8$</th>
<th>$Z_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_3$</td>
<td>0</td>
<td>$2Z_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$Z_4$</td>
<td>$-2Z_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$Z_5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$Z_6$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-2Z_6$</td>
<td>$Z_6$</td>
</tr>
<tr>
<td>$Z_7$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$Z_7$</td>
</tr>
<tr>
<td>$Z_8$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2$Z_6$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$Z_9$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-Z_6$</td>
<td>$-Z_7$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

From the table 3 and the formula (2.29), one acquire the base operators for the adjoint algebra:

\[
A_3 = 2Z_3 \frac{\partial}{\partial Z_4}, \quad A_4 = -2Z_3 \frac{\partial}{\partial Z_3}, \quad A_5 = 0, \quad A_6 = -2Z_6 \frac{\partial}{\partial Z_8} + Z_6 \frac{\partial}{\partial Z_9}, \\
A_7 = Z_7 \frac{\partial}{\partial Z_9}, \quad A_8 = 2Z_6 \frac{\partial}{\partial Z_6}, \quad A_9 = -Z_6 \frac{\partial}{\partial Z_6} - Z_7 \frac{\partial}{\partial Z_7}.
\]

(2.31)

The infinitesimal operator $A_3$ (2.31) is connected to the following one-parameter group of transformations

\[
\begin{align*}
\bar{Z}_3 &= Z_3, & \bar{Z}_4 &= Z_4 + 2a_3 Z_3, & \bar{Z}_5 &= Z_5, \\
\bar{Z}_6 &= Z_6, & \bar{Z}_7 &= Z_7, & \bar{Z}_8 &= Z_8, & \bar{Z}_9 &= Z_9,
\end{align*}
\]

which is represented by the matrix

\[
M_3(a_3) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2a_3 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

By the same procedure we represent all the other transformations, generated by the
operators (2.31), with corresponding matrixes $M_4(a_4) - M_9(a_9)$:

\[
M_4(a_4) = \begin{pmatrix}
1-2a_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad M_5(a_5) = I \quad \text{(The unit matrix)},
\]

\[
M_6(a_6) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -2a_6 & 0 & 1 \\
0 & 0 & a_6 & 0 & 0 & 1 \\
\end{pmatrix}, \quad M_7(a_7) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & a_7 & 1 \\
\end{pmatrix},
\]

\[
M_8(a_8) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad M_9(a_9) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]

A general automorphism will now be obtained by taking the product of these matrixes, $M = M_3(a_3) \cdots M_9(a_9)$:

\[
M = M_3(a_3) \cdots M_9(a_9) = \begin{pmatrix}
1-2a_4 & 0 & 0 & 0 & 0 & 0 \\
2a_3(1-2a_4) & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & (1+2a_8)(1-a_9) & 0 & 0 \\
0 & 0 & 0 & 0 & 1-a_9 & 0 \\
0 & 0 & 0 & -2a_6(1+2a_8)(1-a_9) & 0 & 1 \\
0 & 0 & 0 & a_6(1+2a_8)(1-a_9) & a_7(1-a_9) & 0 \\
\end{pmatrix}.
\]

Considering an arbitrary one-parameter subgroup by a linear combination of the operators (2.30),

\[
Z = \sum_{i=3}^{9} e^{iZ_i}. \quad \text{(2.32)}
\]
Changing $Z$, by an automorphism $A$, will correspond to a changing of the vector

$$e = (e^3, \ldots, e^9)$$

by a linear transformation by means of the transposed matrix $M^T$ of $M$. The image of the vector $e$, under the action of the matrix $M^T$, will give the following new coordinates:

$$\tau^3 = (1 - 2a_4)(e^3 + 2a_3e^4), \quad \tau^4 = e^4, \quad \tau^5 = e^5,$$
$$\tau^6 = (1 + 2a_8)(1 - a_9)(e^6 + a_6(e^9 - 2e^8)), \quad (2.33)$$
$$\tau^7 = (1 - a_9)(e^7 + a_7e^9), \quad \tau^8 = e^8, \quad \tau^9 = e^9.$$

It is clear from (2.33) that the set of nonequivalent operators consist of linear combinations of operators $Z_4, Z_5, Z_8, Z_9$. Namely, the coordinates $\tau^3, \tau^6$ and $\tau^7$ are vanishing for (e.g.) appropriate values on $a_4$, $a_8$ and $a_9$.

Hence, the optimal system of one-dimensional subalgebras of $L_7$ (2.30) are

$$Z^{(1)} = Z_4, \quad Z^{(2)} = Z_5, \quad Z^{(3)} = Z_8, \quad Z^{(4)} = Z_9, \quad Z^{(5)} = Z_4 + \alpha Z_5,$$
$$Z^{(6)} = Z_4 + \alpha Z_8, \quad Z^{(7)} = Z_5 + \alpha Z_9, \quad Z^{(8)} = Z_8 + \alpha Z_9,$$
$$Z^{(9)} = Z_4 + \alpha Z_9, \quad Z^{(10)} = Z_5 + \alpha Z_9, \quad Z^{(11)} = Z_4 + \alpha Z_5 + \beta Z_8,$$
$$Z^{(12)} = Z_4 + \alpha Z_5 + \beta Z_9, \quad Z^{(13)} = Z_4 + \alpha Z_8 + \beta Z_9,$$
$$Z^{(14)} = Z_5 + \alpha Z_8 + \beta Z_9, \quad Z^{(15)} = Z_4 + \alpha Z_5 + \beta Z_8 + \gamma Z_9,$$  

(2.34)

where $\alpha, \beta, \gamma$ are arbitrary constants.

The functions $F(u), G(c, p), H(u, c), N(p)$ for the non-equivalent system, together with their additional admitted operators (beyond the principal Lie algebra $L_P$), are listed in table 4 and 5 (page 166 and 167).

### 2.4 Application

Using the additional operator according to $Z^{(12)}$ in table 5 (page 167), with $\alpha = 1, \beta = 1:

$$X_3 = x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} + 2c \frac{\partial}{\partial c} + p \frac{\partial}{\partial p}.$$ 

It will, together with the principal Lie algebra $L_P$ (2.27), be admitted by systems (2.2) with functions in the form

$$f(u) = K_1 u, \quad g(c, p) = eG\left(\frac{\sqrt{c}}{p}\right), \quad h(u, c) = \sqrt{c} H\left(\frac{\sqrt{c}}{u}\right), \quad n(p) = K_4 p,$$

where $G, H$ are arbitrary functions and $K_1, K_4$ are arbitrary constants.
We consider (e.g.) the system (2.2) with functions
\[ f(u) = K_1 u, \quad g(c, p) = K_2 p \sqrt{c}, \quad h(u, c) = K_3 u, \quad n(p) = K_4 p. \]

Hence, the system (2.2) is written
\[ \frac{\partial u}{\partial t} = 2u - \frac{\partial}{\partial x} \left( u \frac{\partial c}{\partial x} \right), \]
\[ \frac{\partial c}{\partial t} = -K_2 p \sqrt{c}, \]
\[ \frac{\partial p}{\partial t} = K_3 u - K_4 p. \tag{2.35} \]

The latter system (2.35) will admit the operator
\[ X = X_1 + X_2 + X_3 = \frac{\partial}{\partial t} + (x + 1) \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} + 2c \frac{\partial}{\partial c} + p \frac{\partial}{\partial p}, \tag{2.36} \]
where \( X_1 \) and \( X_2 \) are the base operators that belongs to the principal Lie algebra \( L_{\mathcal{P}} \).

Following the same procedure as in the previous chapter, one find that the operator (2.36) has the invariants
\[ \lambda = (x + 1) e^{-t}, \quad \psi_1 = u e^{-t}, \quad \psi_2 = c e^{-2t}, \quad \psi_3 = p e^{-t}. \]

Letting
\[ u = \psi_1(\lambda) e^t, \quad c = \psi_2(\lambda) e^{2t}, \quad p = \psi_3(\lambda) e^t, \quad \lambda = (x + 1) e^{-t} \tag{2.37} \]
and substitute the above equations into the system (2.35):
\[ -\frac{\partial}{\partial \lambda} \left( \lambda \psi_1(\lambda) \right) = -\frac{\partial}{\partial \lambda} \left( \psi_1(\lambda) \psi_2'(\lambda) \right), \tag{2.38} \]
\[ 2\psi_2(\lambda) - \lambda \psi_2'(\lambda) = -K_2 \psi_3(\lambda) \sqrt{\psi_2(\lambda)}, \tag{2.39} \]
\[ \psi_3(\lambda) - \lambda \psi_3'(\lambda) = K_3 \psi_1(\lambda) - K_4 \psi_3(\lambda). \tag{2.40} \]

From the first equation (2.38), it follows
\[ \lambda \psi_1(\lambda) = \psi_1(\lambda) \psi_2'(\lambda) + \alpha, \]
where \( \alpha \) is an arbitrary constant. For simplicity, \( \alpha \) is chosen \( \alpha = 0 \). Thus
\[ \psi_2(\lambda) = \frac{\lambda^2}{2} + \beta \quad (\beta = \text{constant}). \tag{2.41} \]
From equations (2.39) and (2.40), invoking (2.41), one obtain

\[ \psi_3(\lambda) = -\frac{2\beta}{K_2 \sqrt{\lambda^2/2} + \beta} \]  

(2.42)

and

\[ \psi_1(\lambda) = -\frac{(2 + K_4)\lambda^2 + (1 + K_4)2\beta}{K_2 K_3 \left(\lambda^2/2 + \beta\right)^{3/2}} \]  

(2.43)

respectively.

Finally, from equations (2.37), it follows that

\[ u(t, x) = -\frac{(2 + K_4)(x + 1)^2 e^{-2t} + (1 + K_4)2\beta}{K_2 K_3 \left((x + 1)^2 e^{-2t}/2 + \beta\right)^{3/2}}, \]

\[ c(t, x) = \frac{(x + 1)^2}{2} + \beta e^{2t}, \]

\[ p(t, x) = -\frac{4\beta e^t}{K_2 \sqrt{2(x + 1)^2 e^{-2t} + 4\beta}}, \]

where \( \beta \) is an arbitrary constant, is a solution for the system (2.35).
3 Generalized model of tumour growth

3.1 Mathematical model

Again, returning to the model suggested in [1]:

\[
\frac{\partial u}{\partial t} = f(u) - k \frac{\partial}{\partial x} \left( u \frac{\partial c}{\partial x} \right),
\]

\[
\frac{\partial c}{\partial t} = -g(c, p),
\]

\[
\frac{\partial p}{\partial t} = h(u, c) - Kp,
\]

where \( k \) and \( K \) are positive constants.

As mentioned before, the last equation of the above system is written under the assumption that protease decays linearly, with half-life \( K \). We will discard the latter assumption and consider an arbitrary law of the protease decay. Furthermore, we exclude the immaterial constant \( k \), e.g. by stretching \( x \), and assume that the unknown functions, \( f, g, h \) and \( n \), also depend on the independent variables \( t \) and \( x \). So, we will discuss further the following model:

\[
\frac{\partial u}{\partial t} = f(t, x, u) - \frac{\partial}{\partial x} \left( u \frac{\partial c}{\partial x} \right),
\]

\[
\frac{\partial c}{\partial t} = -g(t, x, c, p),
\]

\[
\frac{\partial p}{\partial t} = h(t, x, u, c) - n(t, x, p).
\]  

3.2 Equivalence generator

We consider a continuous local group of equivalence transformations and seek for its generator of the form

\[
Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial c} + \eta^3 \frac{\partial}{\partial p} \\
+ \mu^1 \frac{\partial}{\partial f} + \mu^2 \frac{\partial}{\partial g} + \mu^3 \frac{\partial}{\partial h} + \mu^4 \frac{\partial}{\partial n},
\]

where

\[
\xi = \xi(t, x, u, c, p), \quad \eta = \eta(t, x, u, c, p), \quad \mu = \mu(t, x, u, c, p, f, g, h, n)
\]
The operator (3.3) generates an equivalence group, if and only if it is admitted by the following extended system:

\[
\begin{align*}
    u_t - f + u_x c_x + u c_{xx} &= 0, \quad c_t + g = 0, \quad p_t - h + n = 0, \\
    f_c &= 0, \quad f_p = 0, \quad g_u = 0, \\
    h_p &= 0, \quad n_u = 0, \quad n_c = 0.
\end{align*}
\]

(3.4)

For the invariance test of the system (3.4), we will make use of the prolonged operator (3.3), viz.

\[
\begin{align*}
    \tilde{Y} &= Y + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_3 \frac{\partial}{\partial c_x} + \zeta_4 \frac{\partial}{\partial c_{xx}} + \zeta_5 \frac{\partial}{\partial p_t} \\
    &\quad + \omega_1 \frac{\partial}{\partial f_t} + \omega_2 \frac{\partial}{\partial f_x} + \omega_3 \frac{\partial}{\partial f_c} + \omega_4 \frac{\partial}{\partial f_p} \\
    &\quad + \omega_5 \frac{\partial}{\partial g_t} + \omega_6 \frac{\partial}{\partial g_x} + \omega_7 \frac{\partial}{\partial g_u} + \omega_8 \frac{\partial}{\partial g_c} + \omega_9 \frac{\partial}{\partial g_p} \\
    &\quad + \omega_{10} \frac{\partial}{\partial h_t} + \omega_{11} \frac{\partial}{\partial h_x} + \omega_{12} \frac{\partial}{\partial h_u} + \omega_{13} \frac{\partial}{\partial h_c} + \omega_{14} \frac{\partial}{\partial h_p} \\
    &\quad + \omega_{15} \frac{\partial}{\partial n_t} + \omega_{16} \frac{\partial}{\partial n_x} + \omega_{17} \frac{\partial}{\partial n_u} + \omega_{18} \frac{\partial}{\partial n_c} + \omega_{19} \frac{\partial}{\partial n_p}.
\end{align*}
\]

(3.5)

Using the notation

\[
(x^1, x^2, u^1, u^2, u^3) = (t, x, u, c, p), \quad (f^1, f^2, f^3, f^4) = (f, g, h, n),
\]

\[
(\nu^1, \nu^2, \nu^3, \nu^4, \nu^5) = (\zeta^1, \xi^2, \eta^1, \eta^2, \eta^3)
\]

the prolongation formulae, for the additional coordinates \(\zeta\) and \(\omega\), are written:

\[
\begin{align*}
    \zeta_i^k &= D_i(\eta^k) - u^k D_i(\xi^j), \quad i, j = 1, 2; \quad k = 1, 2, 3; \\
    \zeta_{ij}^k &= D_j(\zeta_i^k) - u^k D_j(\xi^l), \quad i, j, l = 1, 2; \quad k = 1, 2, 3
\end{align*}
\]

(3.6)

and

\[
\begin{align*}
    \omega_{\alpha}^k &= \tilde{D}_\alpha(\mu^k) - f_{\beta}^k \tilde{D}_\alpha(\nu^\beta) \\
    &= \tilde{D}_\alpha(\mu^k) - f_{x}^k \tilde{D}_\alpha(\xi^1) - f_{x}^k \tilde{D}_\alpha(\xi^2) - f_{c}^k \tilde{D}_\alpha(\eta^1) - f_{c}^k \tilde{D}_\alpha(\eta^2) - f_{p}^k \tilde{D}_\alpha(\eta^3), \\
    k &= 1, 2, 3, 4; \quad \alpha = 1, 2, 3, 4, 5.
\end{align*}
\]

(3.7)
Here

\[
\tilde{D}_t = \frac{\partial}{\partial t} + f_t \frac{\partial}{\partial f} + g_t \frac{\partial}{\partial g} + h_t \frac{\partial}{\partial h} + n_t \frac{\partial}{\partial n}
\]
\[
\tilde{D}_x = \frac{\partial}{\partial x} + f_x \frac{\partial}{\partial f} + g_x \frac{\partial}{\partial g} + h_x \frac{\partial}{\partial h} + n_x \frac{\partial}{\partial n}
\]
\[
\tilde{D}_u = \frac{\partial}{\partial u} + f_u \frac{\partial}{\partial f} + h_u \frac{\partial}{\partial h}
\]
\[
\tilde{D}_c = \frac{\partial}{\partial c} + g_c \frac{\partial}{\partial g} + h_c \frac{\partial}{\partial h}
\]
\[
\tilde{D}_p = \frac{\partial}{\partial p} + g_p \frac{\partial}{\partial g} + n_p \frac{\partial}{\partial n}
\]

(3.8)
denote the new total differentiation for the extended system (3.4).

The condition for the operator (3.3) to be an equivalence generator give rise to the following system of determining equations:

\[
\begin{align*}
\zeta_1^1 - \mu^1 + c_x \zeta_2^1 + u_x \zeta_2^2 + c_{xx} \eta^1 + u \zeta_2^2 = 0, \\
\zeta_1^2 + \mu^2 = 0, \\
\zeta_1^3 - \mu^3 + \mu^4 = 0,
\end{align*}
\]

(3.9)

\[
\begin{align*}
\omega_4^1 &= 0, \\
\omega_5^1 &= 0, \\
\omega_6^1 &= 0,
\end{align*}
\]

(3.10)

Let us first solve the equations (3.10). We have

\[
\omega_4^1 \equiv \tilde{D}_v(\mu^1) - f_t \tilde{D}_v(\xi^1) - f_x \tilde{D}_v(\xi^2) - f_u \tilde{D}_v(\eta^1) = \mu_c^1 + g_c \mu_g^1 + h_c \mu_h^1 - f_t \xi_c^1 - f_x \xi_c^2 - f_u \eta_c^1 = 0
\]

and it follows

\[
\mu_c^1 = 0, \\
\mu_g^1 = 0, \\
\mu_h^1 = 0, \\
\xi_c^1 = 0, \\
\xi_c^2 = 0, \\
\eta_c^1 = 0.
\]

(3.11)

Likewise one obtains:

\[
\omega_5^1 \equiv \tilde{D}_p(\mu^1) - f_t \tilde{D}_p(\xi^1) - f_x \tilde{D}_p(\xi^2) - f_u \tilde{D}_p(\eta^1) = \mu_p^1 + g_p \mu_g^1 + h_p \mu_h^1 - f_t \xi_p^1 - f_x \xi_p^2 - f_u \eta_p^1 = 0
\]

\[
\Rightarrow \mu_p^1 = 0, \\
\mu_g^1 = 0, \\
\mu_h^1 = 0, \\
\xi_p^1 = 0, \\
\xi_p^2 = 0, \\
\eta_p^1 = 0
\]

\[
\omega_6^2 \equiv \tilde{D}_a(\mu^2) - g_t \tilde{D}_a(\xi^1) - g_x \tilde{D}_a(\xi^2) - g_u \tilde{D}_a(\eta^1) = \mu_a^2 + f_a \mu_f^2 + h_a \mu_h^2 - g_t \xi_a^1 - g_x \xi_a^2 - g_u \eta_a^1 = 0
\]

\[
\Rightarrow \mu_a^2 = 0, \\
\mu_f^2 = 0, \\
\mu_h^2 = 0, \\
\xi_a^1 = 0, \\
\xi_a^2 = 0, \\
\eta_a^1 = 0
\]
\[ \omega_0^3 \equiv \tilde{D}_p(\mu^3) - h_t \tilde{D}_p(\xi^1) - h_x \tilde{D}_p(\xi^2) - h_u \tilde{D}_p(\eta^1) - h_c \tilde{D}_p(\eta^2) - h_p \tilde{D}_p(\eta^3) \]
\[ = \mu_3^p + g_p \mu_3^g + n_p \mu_3^a - h_c \eta_3^p - h_p \eta_3^p = 0 \]
\[ \Rightarrow \mu_3^p = 0, \quad \mu_3^g = 0, \quad \mu_3^a = 0, \quad \eta_3^p = 0, \]
\[ \omega_3^4 \equiv \tilde{D}_u(\mu^4) - n_t \tilde{D}_u(\xi^1) - n_x \tilde{D}_u(\xi^2) - n_u \tilde{D}_u(\eta^1) - n_c \tilde{D}_u(\eta^2) - n_p \tilde{D}_u(\eta^3) \]
\[ = \mu_4^u + f_u \mu_4^f + h_u \mu_4^h - n_u \eta_4^u = 0 \]
\[ \Rightarrow \mu_4^u = 0, \quad \mu_4^f = 0, \quad \mu_4^h = 0, \quad \eta_4^u = 0, \]
\[ \omega_3^4 \equiv \tilde{D}_c(\mu^4) - n_t \tilde{D}_c(\xi^1) - n_x \tilde{D}_c(\xi^2) - n_u \tilde{D}_c(\eta^1) - n_c \tilde{D}_c(\eta^2) - n_p \tilde{D}_c(\eta^3) \]
\[ = \mu_4^c + g_c \mu_4^g + h_c \mu_4^h - n_c \eta_4^c - n_p \eta_4^p = 0 \]
\[ \Rightarrow \mu_4^c = 0, \quad \mu_4^g = 0, \quad \mu_4^h = 0, \quad \eta_4^c = 0, \quad \eta_4^p = 0. \]

Hence,
\begin{align*}
\xi^1 &= \xi^1(t, x), \quad \xi^2 = \xi^2(t, x), \\
\eta^1 &= \eta^1(t, x, u), \quad \eta^2 = \eta^2(t, x, c), \quad \eta^3 = \eta^3(t, x, p), \quad (3.12) \\
\mu^1 &= \mu^1(t, x, u, f), \quad \mu^2 = \mu^2(t, x, c, p, g, n), \\
\mu^3 &= \mu^3(t, x, u, c, f, h), \quad \mu^4 = \mu^4(t, x, p, n).
\end{align*}

The prolongation formulae (3.6) gives
\begin{align*}
\zeta_1^1 &= \eta_1^1 + u_1 [\eta_1^1 - \xi_1^1] - u_x \xi_1^2 \\
\zeta_1^2 &= \eta_1^2 + c_t [\eta_2^2 - \xi_1^2] - c_x \xi_1^3 \\
\zeta_1^3 &= \eta_1^3 + p_t [\eta_3^3 - \xi_1^3] - p_x \xi_1^4 \quad (3.13)
\end{align*}

and
\[ \zeta_2^2 = \eta_2^2 + c_x [2 \eta_2^x + \xi_2^2 + c_x \eta_2^2] - c_x \xi_2^4 + c_x [\eta_2^2 - 2 \xi_2^2], \quad (3.14) \]

respectively. Substituting the expressions (3.13) and (3.14) into the first equation (3.9) yield:
\begin{align*}
\eta_1^1 + u_t [\eta_1^1 - \xi_1^1] - u_x \xi_1^2 - \mu_1 + c_x [\eta_1^1 + u_x (\eta_1^1 - \xi_1^2)]]
+ u_x [\eta_1^2 + c_x (\eta_2^2 - \xi_2^2)] - c_t \xi_1^1 \\
+ u_2 [\eta_2^2 + c_x (2 \eta_2^x - \xi_2^2) + c_x \eta_2^2] - c_t \xi_2^1 + c_x [\eta_2^2 - 2 \xi_2^2] - 2 c_t \xi_2^1 \\
= 0.
\end{align*}

(3.15)

Singling out terms with \(c_t x\) in (3.15) and equating to zero this terms, we get:
\[-2 u_x \xi_1^1 = 0,\]
whence
\[ \xi_x^1 = 0. \]

Collecting terms with \( c_{xx} \), in (3.15), gives
\[ \eta^1(t, x, u) + u \left[ \xi_t^1(t) + \eta_x^2(t, x, c) - 2\xi_x^2(t, x) - \eta_u^1(t, x, u) \right] = 0. \tag{3.16} \]

Since \( \eta^2 \) is the only function in (3.16) which depend on \( c \), it follows
\[ \eta^2 = c \varphi^1(t, x) + \varphi^2(t, x), \tag{3.17} \]

where \( \varphi^1 \) and \( \varphi^2 \) are arbitrary functions.

Continuing with terms which depends on \( c_x \), we obtain
\[ u_x \left[ \xi_t^1 - 2\xi_x^2 + \eta_c^2 \right] + \eta_x^1 + u \left[ 2\eta_x^2 - \xi_x^2 \right] = 0. \tag{3.18} \]

The terms with \( u_x \) in (3.18) gives
\[ \xi_t^1 - 2\xi_x^2 + \eta_c^2 = 0 \tag{3.19} \]

and the equation (3.16) is rewritten
\[ \eta^1(t, x, u) - u \eta_u^1(t, x, u) = 0. \tag{3.20} \]

Thus
\[ \eta^1 = u \psi(t, x), \tag{3.21} \]

where \( \psi \) is an arbitrary function of \( t \) and \( x \). From the equation (3.18), invoking equations (3.17), (3.19) and (3.21), we also have
\[ u \psi_x(t, x) + u \left[ 2\varphi_x^1(t, x) - \xi_x^2 \right] = 0. \tag{3.22} \]

Further, collecting terms with \( u_x \) in (3.15):
\[ -\xi_t^2 + \eta_x^2 = 0 \tag{3.23} \]

or, due to the equation (3.17),
\[ -\xi_t^2 + c \varphi_x^1(t, x) + \varphi_x^2(t, x) = 0. \tag{3.24} \]

Hence
\[ \varphi^1 = \varphi^1(t) \implies \eta^2 = c \varphi^1(t) + \varphi^2(t, x) \tag{3.25} \]

and
\[ \xi_t^2 = \varphi_x^2(t, x). \tag{3.26} \]
Now, substitute the expression for $\xi^2$ (3.25) into (3.19):

$$\xi^1_t(t) - 2\xi^2_x(t, x) + \varphi^1(t) = 0.$$  

Thus

$$\xi^2 = \frac{x}{2} \left[ \xi^1_t(t) + \varphi^1(t) \right] + \alpha(t), \quad (3.27)$$

where $\alpha$ is arbitrary. The expression for $\xi^2$ (3.27), $\eta^1$ (3.25) and equation (3.22) gives

$$\eta^1 = u \psi(t). \quad (3.28)$$

The equations (3.24) and (3.27) yield an expression for $\eta^2$:

$$\eta^2 = c \varphi^1(t) + \frac{x^2}{4} \left[ \xi^1_{tt}(t) + \varphi^1_t(t) \right] + x \alpha_t(t) + \beta(t). \quad (3.29)$$

The remaining terms in the first determining equation (3.15), with the substitutions according to equations (3.4), gives

$$\eta^1_t + f \left[ \eta^1_u - \xi^1_t \right] - \mu^1(t, x, u, f) + u \eta^2_{xx} = 0$$

or

$$\mu^1 = f \left[ \psi(t) - \xi^1_t(t) \right] + u \psi_t(t). \quad (3.30)$$

From the prolongation formulae (3.13), it follows that the second determining equation (3.9), $\zeta^2_t + \mu^2 = 0$, can be rewritten:

$$\eta^2_t + c_t \eta^2_c - c_t \xi^1_t - c_x \xi^2_t + \mu^2 = 0. \quad (3.31)$$

Collecting terms with $c_x$ from the above equation gives

$$\xi^2_t = 0 \quad \text{or} \quad \frac{x}{2} \left[ \xi^1_{tt} + \varphi^1_t(t) \right] + \alpha(t) = 0. \quad (3.32)$$

Hence

$$\alpha(t) = K_1 \quad \text{and} \quad \varphi^1(t) = K_2 - \xi^1_t(t), \quad (3.33)$$

where $K_1$ and $K_2$ are constants.

Having in mind the condition $c_t + g = 0$ (3.4) and equation (3.32), the equation (3.31) is written

$$\eta^2_t + g \left[ \xi^1_t - \eta^2_c \right] + \mu^2(t, x, c, p, g, n) = 0$$

or, invoking (3.23) and (3.29),

$$\mu^2 = g \left[ K_2 - 2 \xi^1_t \right] + c \xi^1_{tt}(t) - \beta_t(t). \quad (3.34)$$
Finally, the third determining equation (3.9), \( \zeta_1^3 - \mu^3 + \mu^4 = 0 \), gives
\[
\eta^3(t, x, p) + \left( h - n \right) \eta^3_p(t, x, p) - \left( h - n \right) \xi^1_t(t) - \mu^3(t, x, u, c, f, h) - \mu^4(t, x, p, n) = 0. \tag{3.35}
\]

By differentiating the above equation with respect to \( h \),
\[
\eta^3_p - \xi^1_t - \mu^3_p = 0 \tag{3.36}
\]
and \( p \), it follows
\[
\eta^3_{pp} = 0.
\]
Thus
\[
\eta^3 = p \lambda(t, x) + \sigma(t, x), \tag{3.37}
\]
where \( \lambda \) and \( \sigma \) are arbitrary functions of \( t \) and \( x \). Now, equation (3.36) yield:
\[
\mu^3 = h \left[ \lambda(t, x) - \xi^1_t(t) \right] + \tau(t, x), \tag{3.38}
\]
where \( \tau \) is an arbitrary function. \( \mu^4 \) is obtained from (3.35):
\[
\mu^4 = n \left[ \lambda(t, x) - \xi^1_t(t) \right] - p \lambda_t(t, x) - \sigma_t(t, x) + \tau(t, x). \tag{3.39}
\]

Finally, collecting all coefficients form (3.27)–(3.30), (3.34) and (3.37)–(3.39), we have:
\[
\xi^1 = \xi(t), \quad \xi^2 = \frac{\xi}{2} K_2 + K_1, \quad \eta^1 = u \psi(t), \quad \eta^2 = c \left[ K_2 - \xi_t(t) \right] + \beta(t),
\eta^3 = p \lambda(t, x) + \sigma(t, x), \quad \mu^1 = f \left[ \psi(t) - \xi_t(t) \right] + u \psi_t(t),
\mu^2 = g \left[ K_2 - 2 \xi_t \right] + c \xi_{tt}(t) - \beta_t(t), \quad \mu^3 = h \left[ \lambda(t, x) - \xi_t(t) \right] + \tau(t, x),
\mu^4 = n \left[ \lambda(t, x) - \xi_t(t) \right] - p \lambda_t(t, x) - \sigma_t(t, x) + \tau(t, x),
\]
where \( \xi, \psi, \beta, \lambda, \sigma, \tau \) are arbitrary functions. \( K_1 \) and \( K_2 \) are arbitrary constants.

The infinitesimal generators of the equivalence group generates an infinite Lie group \( L_\infty \):
\[
Y_1 = \frac{\partial}{\partial x}; \quad Y_2 = x \frac{\partial}{\partial x} + 2c \frac{\partial}{\partial c} + 2g \frac{\partial}{\partial g}; \quad Y_\psi = u \psi(t) \frac{\partial}{\partial u} + \left[ f \psi(t) + u \psi_t(t) \right] \frac{\partial}{\partial f},
Y_\xi = \xi(t) \frac{\partial}{\partial t} - c \xi_t(t) \frac{\partial}{\partial c} - f \xi_t(t) \frac{\partial}{\partial f} - \left[ 2g \xi_t(t) - c \xi_{tt}(t) \right] \frac{\partial}{\partial g} - h \xi_t(t) \frac{\partial}{\partial h} - n \xi_t(t) \frac{\partial}{\partial n},
Y_\beta = \beta(t) \frac{\partial}{\partial c} - \beta_t(t) \frac{\partial}{\partial g}; \quad Y_\lambda = p \lambda(t, x) \frac{\partial}{\partial p} + h \lambda(t, x) \frac{\partial}{\partial h} + \left[ n \lambda(t, x) - p \lambda_t(t, x) \right] \frac{\partial}{\partial n},
Y_\sigma = \sigma(t, x) \frac{\partial}{\partial p} - \sigma_t(t, x) \frac{\partial}{\partial n}; \quad Y_\tau = \tau(t, x) \frac{\partial}{\partial h} + \tau(t, x) \frac{\partial}{\partial n}. \tag{3.41}
\]
The Lie group $L_\infty$ gives the following table of commutators, $[X, Y] = XY - YX$.

In table 6, the operators which have indices with tilde are in the same form as their corresponding “originating” operators in $L_\infty$ (e.g. $Y_\tilde{\xi}$ are in the same form as $Y_\psi$). The functions will appear as

$$
\begin{align*}
\tilde{\lambda}_1(t, x) &= \lambda_x(t, x), & \tilde{\sigma}_1(t, x) &= \sigma_x(t, x), & \tilde{\tau}_1(t, x) &= \tau_x(t, x), \\
\tilde{\lambda}_2(t, x) &= x\lambda_x(t, x), & \tilde{\sigma}_2(t, x) &= x\sigma_x(t, x), & \tilde{\tau}_2(t, x) &= x\tau_x(t, x), \\
\tilde{\psi}_\tilde{\xi}(t) &= \xi(t)\psi_\xi(t), & \tilde{\beta}_\tilde{\xi}(t) &= \xi(t)\beta_\xi(t) + \xi_\xi(t)\beta(t), & \tilde{\lambda}_\tilde{\xi}(t, x) &= \xi(t)\lambda_\xi(t, x), \\
\tilde{\sigma}_\tilde{\xi}(t, x) &= \xi(t)\sigma_\xi(t, x), & \tilde{\tau}_\tilde{\xi}(t, x) &= \xi(t)\tau_\xi(t, x) + \xi_\xi(t)\tau(t, x), \\
\tilde{\sigma}_\lambda(t, x) &= -\lambda(t, x)\sigma(t, x), & \tilde{\tau}_\lambda(t, x) &= -\lambda(t, x)\tau(t, x).
\end{align*}
$$

(3.42)

### 3.3 Application

To find invariant solutions of the system (3.2), we use subgroups of the infinite Lie algebra $L_\infty$ (3.41), e.g.:

#### 3.3.1 Example 1

Consider the two-dimensional subgroup spanned by $Y_1$ and $Y_A = Y_\xi + Y_\psi + Y_\sigma$ ($A_1, Y_\xi, Y_\psi, Y_\sigma \in L_\infty$ (3.41)), where

$$\xi = t^2 + 1, \quad \psi = 1, \quad \sigma = t.$$

Hence

$$
\begin{align*}
Y_1 &= \frac{\partial}{\partial x}, \\
Y_A &= (t^2 + 1)\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} - 2tc\frac{\partial}{\partial c} + t\frac{\partial}{\partial p} + (1 - 2t)f\frac{\partial}{\partial f} - (4tg - 2c)\frac{\partial}{\partial g} \\
&\quad - 2th\frac{\partial}{\partial h} - (2tn + 1)\frac{\partial}{\partial n},
\end{align*}
$$

(3.43)

with the commutator $[Y_1, Y_A] = 0$.

The condition for the system to admit the subgroup (3.43), projected on $t, x, u, c, p$, is that the functions

$$f = F(t, x, u), \quad g = G(t, x, c, p), \quad h = H(t, x, u, c), \quad n = N(t, x, p), \quad (3.44)$$


admit the operators $Y_1$ and $Y_A$, projected on the arguments of $f$, $g$, $h$, $n$. The invariance conditions for the equations (3.44) will give rise to the following equations

$$
\begin{align*}
\text{pr}_{(t,x,u)}(Y_A) \left( f - F \right) \bigg|_{f=F(t,x,u)} &\equiv (1 - 2t)F - (t^2 + 1)F_t - uF_u = 0, \\
\text{pr}_{(t,x,c,p)}(Y_A) \left( g - G \right) \bigg|_{g=G(t,x,c,p)} &\equiv -(4tG - 2c) - (t^2 + 1)G_t + 2ctG_c - tG_p = 0, \\
\text{pr}_{(t,x,u,c)}(Y_A) \left( h - H \right) \bigg|_{h=H(t,x,u,c)} &\equiv -2tH - (t^2 + 1)H_t - uH_u + 2tcH_c = 0, \\
\text{pr}_{(t,x,p)}(Y_A) \left( n - N \right) \bigg|_{n=N(t,x,p)} &\equiv -(2tN + 1) - (t^2 + 1)N_t - tN_p = 0,
\end{align*}
(3.45)

and

$$
\begin{align*}
\text{pr}_{(t,x,u)}(Y_1) \left( f - F \right) \bigg|_{f=F(t,x,u)} &\equiv F_x = 0, \\
\text{pr}_{(t,x,c,p)}(Y_1) \left( g - G \right) \bigg|_{g=G(t,x,c,p)} &\equiv G_x = 0, \\
\text{pr}_{(t,x,u,c)}(Y_1) \left( h - H \right) \bigg|_{h=H(t,x,u,c)} &\equiv H_x = 0, \\
\text{pr}_{(t,x,p)}(Y_1) \left( n - N \right) \bigg|_{n=N(t,x,p)} &\equiv N_x = 0,
\end{align*}
(3.46)

whence

$$
\begin{align*}
f &= \frac{\arctan t}{t^2 + 1} F_1\left(ue^{-\arctan t}\right), \\
g &= \frac{2tc}{t^2 + 1} + \frac{c^2 G_1((t^2 + 1)c, \sqrt{t^2 + 1} e^{-p})}{(t^2 + 1)^{3/2}}, \\
h &= cH_1\left(ue^{-\arctan t}, (t^2 + 1)c\right), \\
n &= \frac{-t}{t^2 + 1} + e^{-2p} N_1\left(\sqrt{t^2 + 1} e^{-p}\right),
\end{align*}
$$

where $F_1, G_1, H_1, N_1$ are arbitrary functions.

Letting (e.g.)

$$
\begin{align*}
f &= \frac{u}{t^2 + 1}, \\
g &= \frac{2tc}{t^2 + 1} + \frac{c e^{-p}}{\sqrt{t^2 + 1}}, \\
h &= u e^{-\arctan t}, \\
n &= \frac{-t}{t^2 + 1} + \frac{e^p}{(t^2 + 1)^{3/2}},
\end{align*}
$$

the system (3.2) will be written in the form

$$
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{u}{t^2 + 1} - \frac{\partial u}{\partial x} \left( \frac{\partial c}{\partial x} \right), \\
\frac{\partial c}{\partial t} &= -\frac{2tc}{t^2 + 1} - \frac{c e^{-p}}{\sqrt{t^2 + 1}}, \\
\frac{\partial p}{\partial t} &= u e^{-\arctan t} + \frac{t}{t^2 + 1} - \frac{e^p}{(t^2 + 1)^{3/2}}.
\end{align*}
(3.47)
Now, the equations (3.47) admit the operators (3.43) projected on \((x, u) = (t, x, u, c, p)\). For example, let’s consider the sum of the operators (3.43):

\[
X = \text{pr}_{(x,u)}(Y_1 + Y_A) = (t^2 + 1) \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} - 2tc \frac{\partial}{\partial c} + t \frac{\partial}{\partial p},
\]

which has the invariants

\[
x - \arctan t, \quad \psi_1 = u e^{-\arctan t}, \quad \psi_2 = (t^2 + 1)c, \quad \psi^3 = \sqrt{t^2 + 1} e^{-p}.
\]

It follows that the invariant solutions of \(u(t, x), c(t, x), p(t, x)\) will have the form

\[
u = \psi_1(y)e^{\arctan t}, \quad c = \frac{\psi_2(y)}{t^2 + 1}, \quad p = \ln \left(\sqrt{t^2 + 1} - \ln \psi_3(y)\right),
\]

where \(y = x - \arctan t\). Substituting the above expressions (3.48) into the system (3.47) one obtain a system of ordinary differential equations

\[
\begin{align*}
\psi'_1(y) &= \frac{d}{dy} \left(\psi_1(y) \psi'_2(y)\right), \\
\psi'_2(y) &= \psi_2(y) \psi_3(y), \\
\psi'_3(y) &= \psi_1(y) \psi_2(y) \psi_3(y) - 1.
\end{align*}
\]

By integrate the equation (3.49) with respect to \(y\):

\[
\psi_1(y) = \psi_1(y) \psi'_2(y) + A \quad (A = \text{const.})
\]

and consider the simple case when the arbitrary constant \(A\) is equal to zero, one obtain a solution for (3.49):

\[
\psi_1(y) = 1 - \frac{1}{(y + \alpha)^2}, \quad \psi_2(y) = y + \alpha, \quad \psi_3(y) = \frac{1}{y + \alpha},
\]

where \(\alpha\) is an arbitrary constant.

Substituting the above equations into (3.48), with \(y = x - \arctan t\), gives one invariant solution for the system (3.47).

\[
u(t, x) = \left(1 - \frac{1}{(x - \arctan t + \alpha)^2}\right) e^{\arctan t}, \quad (3.52)
\]

\[
c(t, x) = \frac{x - \arctan t + \alpha}{t^2 + 1}, \quad (3.53)
\]

\[
p(t, x) = \ln \left( (x - \arctan t + \alpha) \sqrt{t^2 + 1} \right) \quad (3.54)
\]

The functions (3.52)–(3.54) are illustrated in the figures below.
Here, the figure to the left illustrate the function $u(t, x)$ according to (3.52), with $\alpha = 3$.

The function $c(t, x)$ according to (3.53), with $\alpha = 3$.

The function $p(t, x)$ according to (3.54), with $\alpha = 3$. 
3.3.2 Example 2

As a second example, we consider the two-dimensional subgroup spanned by $Y_1$ and $Y_A = Y_\xi + Y_\psi + Y_\lambda$ ($A_1, Y_\xi, Y_\psi, Y_\lambda \in L_\infty$ (3.41)), where

$$
\xi = t^2 + 1, \quad \psi = t, \quad \lambda = 2t, \quad ([Y_1, Y_A] = 0).
$$

Hence

$$
Y_1 = \frac{\partial}{\partial x},
Y_A = (t^2 + 1) \frac{\partial}{\partial t} + tu \frac{\partial}{\partial u} - 2tc \frac{\partial}{\partial c} + 2tp \frac{\partial}{\partial p} + (u - tf) \frac{\partial}{\partial f} \quad (3.55)
$$

$$
-(4tg - 2c) \frac{\partial}{\partial g} - 2p \frac{\partial}{\partial n}.
$$

The invariance conditions for the arbitrary functions,

$$
f = F(t, x, u), \quad g = G(t, x, c, p), \quad h = H(t, x, u, c), \quad n = N(t, x, p), \quad (3.56)
$$

are expressed by the following partial differential equations:

$$
\begin{align*}
\text{pr}_{(t,x,u)}(Y_A) (f - F) \big|_{f=F(t,x,u)} &\equiv u - tF - (t^2 + 1)F_t - tuF_u = 0, \\
\text{pr}_{(t,x,c,p)}(Y_A) (g - G) \big|_{g=G(t,x,c,p)} &\equiv -(4tG - 2c) - (t^2 + 1)G_t + 2ctG_c - 2tpG_p = 0, \\
\text{pr}_{(t,x,u,c)}(Y_A) (h - H) \big|_{h=H(t,x,u,c)} &\equiv -(t^2 + 1)H_t - tuH_u + 2tcH_c = 0, \\
\text{pr}_{(t,x,p)}(Y_A) (n - N) \big|_{n=N(t,x,p)} &\equiv -2p - (t^2 + 1)N_t - 2tpN_p = 0,
\end{align*}
\quad (3.57)
$$

and

$$
\begin{align*}
\text{pr}_{(t,x,u)}(Y_1) (f - F) \big|_{f=F(t,x,u)} &\equiv F_x = 0, \\
\text{pr}_{(t,x,c,p)}(Y_1) (g - G) \big|_{g=G(t,x,c,p)} &\equiv G_x = 0, \\
\text{pr}_{(t,x,u,c)}(Y_1) (h - H) \big|_{h=H(t,x,u,c)} &\equiv H_x = 0, \\
\text{pr}_{(t,x,p)}(Y_1) (n - N) \big|_{n=N(t,x,p)} &\equiv N_x = 0,
\end{align*}
\quad (3.58)
$$

whence

$$
\begin{align*}
f &= \frac{ut}{t^2 + 1} + \frac{1}{\sqrt{t^2 + 1}} F_1 \left( \frac{u}{\sqrt{t^2 + 1}} \right), \\
g &= \frac{2tc}{t^2 + 1} + c^2 G_1 \left( (t^2 + 1)c, \frac{p}{t^2 + 1} \right), \\
h &= H_1 \left( \frac{u}{\sqrt{t^2 + 1}}, (t^2 + 1)c \right), \\
n &= -\frac{2tc}{t^2 + 1} + N_1 \left( \frac{p}{t^2 + 1} \right),
\end{align*}
$$
where $F_1, G_1, H_1, N_1$ are arbitrary functions.

Letting (e.g.)

\[
\begin{align*}
    f &= \frac{ut}{t^2 + 1}, \quad g = \frac{2tc}{t^2 + 1} + \frac{c^2p}{t^2 + 1}, \\
    h &= uc\sqrt{t^2 + 1}, \quad n = \frac{p^2}{(t^2 + 1)^2} - \frac{2tp}{t^2 + 1}.
\end{align*}
\]

The system (3.2) is rewritten

\[
\begin{align*}
    \frac{\partial u}{\partial t} &= \frac{ut}{t^2 + 1} - \frac{\partial}{\partial x}\left(u\frac{\partial c}{\partial x}\right), \\
    \frac{\partial c}{\partial t} &= -\frac{2tc + c^2p}{t^2 + 1}, \\
    \frac{\partial p}{\partial t} &= uc\sqrt{t^2 + 1} + \frac{2tp}{t^2 + 1} - \frac{p^2}{(t^2 + 1)^2}.
\end{align*}
\] (3.59)

The equations (3.59) is now admitting the operators (3.55) projected on $(x, u, c, p)$:

\[
\text{pr}_{(x, u)}(Y_1 + Y_A) = (t^2 + 1)\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + tu\frac{\partial}{\partial u} - 2tc\frac{\partial}{\partial c} + 2tp\frac{\partial}{\partial p},
\]

which has the invariants

\[
x - \arctan t, \quad \psi_1 = \frac{u}{\sqrt{t^2 + 1}}, \quad \psi_2 = (t^2 + 1)c, \quad \psi_3 = \frac{p}{t^2 + 1}.
\]

It follows that $u(t, x), c(t, x), p(t, x)$ are written

\[
\begin{align*}
    u &= \psi_1(y)\sqrt{t^2 + 1}, \quad c = \frac{\psi_2(y)}{t^2 + 1}, \quad p = (t^2 + 1)\psi_3(y),
\end{align*}
\] (3.60)

where $y = x - \arctan t$. Substituting the above expression (3.60) into the system (3.59), one obtain a system of ordinary differential equations:

\[
\begin{align*}
    \psi_1'(y) &= \frac{d}{dy}\left(\psi_1(y)\psi_2'(y)\right), \quad (3.61) \\
    \psi_2'(y) &= \psi_3(y), \quad (3.62) \\
    \psi_3'(y) &= -\psi_1(y)\psi_2(y) + \psi_3(y). \quad (3.63)
\end{align*}
\]
By integrating the equation (3.61) with respect to $y$

$$
\psi_1(y) = \psi_1(y)\psi'_2(y) + A \quad (A = \text{const.}),
$$
and again considering the simple case when $A = 0$, one obtain a solution given by

$$
\psi_1(y) = \frac{2(y + \alpha) + 1}{(y + \alpha)^5}, \quad \psi_2(y) = y + \alpha, \quad \psi_3(y) = \frac{1}{(y + \alpha)^2},
$$

where $\alpha$ is an arbitrary constant.

Substituting the above equations into (3.60), with $y = x - \arctan t$, gives one invariant solution for the system (3.59).

$$
\begin{align*}
  u(t, x) &= \frac{\left[2 \left(x - \arctan t + \alpha\right) + 1\right] \sqrt{t^2 + 1}}{\left(x - \arctan t + \alpha\right)^5}, \\
  c(t, x) &= \frac{x - \arctan t + \alpha}{t^2 + 1}, \\
  p(t, x) &= \frac{t^2 + 1}{\left(x - \arctan t + \alpha\right)^2}.
\end{align*}
$$

(3.64) (3.65) (3.66)

The functions (3.64)–(3.66) are illustrated in the figures below.
Here, the figure to the left illustrate the function $u(t, x)$ according to (3.64), with $\alpha = 3$.

The function $c(t, x)$ according to (3.65), with $\alpha = 3$.

The function $p(t, x)$ according to (3.66), with $\alpha = 3$. 
Acknowledgements

I would like to express my gratitude to my supervisor professor Nail H. Ibragimov, who introduce me into the interesting field of group analysis. His help and encouragement during my work has been most valuable.
Table 4: Optimal system for the system of equations (2.2). $\alpha, \beta, \gamma$ and $K_1, K_4$ are arbitrary constants. $F, G, H, N$ are arbitrary functions.

<table>
<thead>
<tr>
<th>$Z$</th>
<th>Functions</th>
<th>Additional operator $X_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z^{(1)}$</td>
<td>$f = F(u)$, $g = c G(p)$, $h = H(u)$, $n = N(p)$</td>
<td>$x \frac{\partial}{\partial x} + 2c \frac{\partial}{\partial c}$</td>
</tr>
<tr>
<td>$Z^{(2)}$</td>
<td>$f = K_1 u$, $g = G(c, p)$, $h = H(c)$, $n = N(p)$</td>
<td>$u \frac{\partial}{\partial u}$</td>
</tr>
<tr>
<td>$Z^{(3)}$</td>
<td>$f = g = h = n = 0$</td>
<td>$2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}$</td>
</tr>
<tr>
<td>$Z^{(4)}$</td>
<td>$f = F(u)$, $g = G(c)$, $h = 0$, $n = K_4 p$</td>
<td>$p \frac{\partial}{\partial p}$</td>
</tr>
<tr>
<td>$Z^{(5)}$</td>
<td>$f = K_1 u$, $g = c G(p)$, $h = H(c^{\alpha/2}/u)$, $n = N(p)$</td>
<td>$x \frac{\partial}{\partial x} + \alpha u \frac{\partial}{\partial u} + 2c \frac{\partial}{\partial c}$</td>
</tr>
<tr>
<td>$Z^{(6)}$</td>
<td>$f = 0$, $g = c^{(1-\alpha)} G(p)$, $h = c^{-\alpha} H(u)$, $n = 0$</td>
<td>$2\alpha t \frac{\partial}{\partial t} + (1 + \alpha)x \frac{\partial}{\partial x} + 2c \frac{\partial}{\partial c}$</td>
</tr>
<tr>
<td>$Z^{(7)}$</td>
<td>$f = K_1 u^{(1-2\alpha)}$, $g = 0$, $h = u^{-2\alpha} H(c)$, $n = 0$</td>
<td>$2\alpha t \frac{\partial}{\partial t} + \alpha x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}$</td>
</tr>
<tr>
<td>$Z^{(8)}$</td>
<td>$f = 0$, $g = p^{-2/\alpha} G(c)$, $h = 0$, $n = K_4 p^{(1-2/\alpha)}$</td>
<td>$2t \frac{\partial}{\partial t} + \frac{x}{\alpha} \frac{\partial}{\partial x} + \frac{\partial}{\partial p}$</td>
</tr>
<tr>
<td>$Z^{(9)}$</td>
<td>$f = F(u)$, $g = c G(c^{\alpha/2}/p)$, $h = c^{\alpha/2} H(u)$, $n = K_4 p$</td>
<td>$x \frac{\partial}{\partial x} + 2c \frac{\partial}{\partial c} + \alpha p \frac{\partial}{\partial p}$</td>
</tr>
<tr>
<td>$Z^{(10)}$</td>
<td>$f = K_1 u$, $g = G(c)$, $h = u^{\alpha} H(c)$, $n = K_4 p$</td>
<td>$u \frac{\partial}{\partial u} + \alpha p \frac{\partial}{\partial p}$</td>
</tr>
<tr>
<td>$Z^{(11)}$</td>
<td>$f = K_1 u^{(1-2\beta/\alpha)}$, $g = c^{(1-\beta)} G(p)$, $h = c^{-\beta} H(c^{\alpha/2}/u)$, $n = 0$</td>
<td>$2\beta t \frac{\partial}{\partial t} + (1 + \beta)x \frac{\partial}{\partial x} + \alpha u \frac{\partial}{\partial u} + 2c \frac{\partial}{\partial c}$</td>
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</tbody>
</table>
Table 5: (Continuation of table 4) Optimal system for the system of equations (2.2). $\alpha$, $\beta$, $\gamma$ and $K_1$, $K_4$ are arbitrary constants. $F, G, H, N$ are arbitrary functions.

<table>
<thead>
<tr>
<th>$Z$</th>
<th>Functions</th>
<th>Additional operator $X_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z^{(12)}$</td>
<td>$f = K_1 u,$ $g = c G(c^{\beta/2}/p),$ $h = c^{\beta/2} H(c^{\alpha/2}/u),$ $n = K_4 p$</td>
<td>$x \frac{\partial}{\partial x} + \alpha u \frac{\partial}{\partial u} + 2c \frac{\partial}{\partial c} + \beta p \frac{\partial}{\partial p}$</td>
</tr>
<tr>
<td>$Z^{(13)}$</td>
<td>$f = 0,$ $g = c^{(1-\alpha)} G(c^{\beta/2}/p),$ $h = c^{(\beta/2-\alpha)} H(u),$ $n = K_4 p^{(1-2\alpha/\beta)}$</td>
<td>$2\alpha t \frac{\partial}{\partial t} + (1 + \alpha) x \frac{\partial}{\partial x} + 2c \frac{\partial}{\partial c} + \beta p \frac{\partial}{\partial p}$</td>
</tr>
<tr>
<td>$Z^{(14)}$</td>
<td>$f = K_1 u^{(1-2\alpha)},$ $g = p^{-2\alpha/\beta} G(c),$ $h = u^{(\beta-2\alpha)} H(c),$ $n = K_4 p^{(1-2\alpha/\beta)}$</td>
<td>$2\alpha t \frac{\partial}{\partial t} + \alpha x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} + \beta p \frac{\partial}{\partial p}$</td>
</tr>
<tr>
<td>$Z^{(15)}$</td>
<td>$f = K_1 u^{(1-2\beta/\alpha)},$ $g = c^{(1-\beta)} G(c^{\gamma/2}/p),$ $h = c^{(\gamma/2-\beta)} H(c^{\alpha/2}/u),$ $n = K_4 p^{(1-2\beta/\gamma)}$</td>
<td>$2\beta t \frac{\partial}{\partial t} + (1 + \beta) x \frac{\partial}{\partial x} + \alpha u \frac{\partial}{\partial u} + 2c \frac{\partial}{\partial c} + \gamma p \frac{\partial}{\partial p}$</td>
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Table 6: Table of commutators, $L_{\infty}$.

<table>
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<tr>
<th>$[r, c]$</th>
<th>$Y_1$</th>
<th>$Y_2$</th>
<th>$Y_\xi$</th>
<th>$Y_\psi$</th>
<th>$Y_\beta$</th>
<th>$Y_\lambda$</th>
<th>$Y_\sigma$</th>
<th>$Y_\tau$</th>
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</thead>
<tbody>
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<td>$Y_1$</td>
<td>0</td>
<td>$Y_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$Y_\lambda^1$</td>
<td>$Y_\sigma^1$</td>
<td>$Y_\tau^1$</td>
</tr>
<tr>
<td>$Y_2$</td>
<td>$-Y_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-2Y_\beta$</td>
<td>$Y_\lambda^2$</td>
<td>$Y_\sigma^2$</td>
</tr>
<tr>
<td>$Y_\xi$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$Y_\psi^\xi$</td>
<td>0</td>
<td>$Y_\lambda^\xi$</td>
<td>$Y_\sigma^\xi$</td>
<td>$Y_\tau^\xi$</td>
</tr>
<tr>
<td>$Y_\psi$</td>
<td>0</td>
<td>0</td>
<td>$-Y_\psi^\xi$</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
</tr>
<tr>
<td>$Y_\beta$</td>
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<td>0</td>
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<td>0</td>
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<tr>
<td>$Y_\lambda$</td>
<td>$-Y_\lambda^1$</td>
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<td>$Y_\tau^\lambda$</td>
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<td>0</td>
<td>$-Y_\tau^\lambda$</td>
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</tr>
</tbody>
</table>
Secondary bibliography


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Archives of ALGA

Brief facts about the centre ALGA: Advances in Lie Group Analysis

ALGA at Blekinge Institute of Technology, Sweden, is an international research and educational centre aimed at producing new knowledge of Lie group analysis of differential equations and enhancing the understanding of the classical results and modern developments.

The main objectives of ALGA are:

- To make available to a wide audience the classical heritage in group analysis and to teach courses in Lie group analysis as well as new mathematical programs based on the philosophy of group analysis.
- To advance studies in modern group analysis, differential equations and nonlinear mathematical modelling and to implement a database containing all the latest information in this field.

For more information, contact the director of ALGA, Professor N.H. Ibragimov.
E-mail: nib@bth.se
Homepage: www.bth.se/alga
Address: ALGA, Blekinge Institute of Technology, S-371 79 Karlskrona, Sweden.

Aims and Scope of Archives

Aim: The aim of the Archives is to provide an international forum for classical and modern group analysis by means of rapid communication of new results, review articles and publications of historical heritage. The publications are related to the activities of ALGA.

Scope: The scope of Archives encompasses Lie group analysis with a focus on nonlinear differential equations and mathematical models in science and engineering.