Archives of ALGA

Volume 1, July 2004

• Equivalence groups and invariants
• Linearization of third-order equations
• Group analysis of stochastic differential systems

ALGA Publications
BLEKINGE INSTITUTE OF TECHNOLOGY
Karlskrona, Sweden
Archives of ALGA

Revised edition

ALGA Publications
Blekinge Institute of Technology
Karlskrona, Sweden
Information for authors

The manuscript should be prepared in LaTeX using only the standard LaTeX commands. The manuscripts should be sent to nib@bth.se.

1. The author assigns to ALGA the exclusive right to reproduce the contribution in Archives of ALGA and to distribute it both in printed and electronic versions. The latter includes publication at the ALGA website.

2. The author guarantees that the contribution is original and has not been published previously or been submitted for publication elsewhere in the same form.

3. The author retains the right to republish the contribution in whole or in part in his further research, education and for other purposes without asking written permission from ALGA provided the author acknowledges the original article in Archives of ALGA.

4. The copyright to the whole volume of Archives belongs solely to ALGA.

© 2004 ALGA

All rights reserved. No part of this publication may be reproduced or utilized in any form or by any means, electronic or mechanical, including photocopying, recording, scanning or otherwise, without written permission from ALGA.
Preface

Lie group analysis is a growing field of mathematics with numerous applications. This is due to the fact that ideas of symmetry and invariance permeate all mathematical models in natural and engineering sciences. Nowadays, an acquaintance with the classical foundations and modern methods of Lie group analysis has become an important part of the mathematical culture of anyone constructing and/or investigating non-linear mathematical models. Moreover, Lie’s original approach has been extended to new situations and has become applicable to the majority of differential, integro-differential and stochastic differential equations, which frequently occur in applications. However, no research or education institute existed focused specifically on this field. ALGA has been instituted to fill this gap.

Landmarks

- **1983** - A research laboratory for Lie group analysis was founded by N.H. Ibragimov at Aviation Technical University, Ufa, Russia
- **1987** - Continued at the Institute of Applied Mathematics and then at the Institute of Mathematical Modelling of the USSR Academy of Sciences in Moscow
- **1994** - International status is achieved as Centre for Symmetry Analysis and Differential Equations at the University of the Witwatersrand, Johannesburg, South Africa
- **1998** - Continued as International Institute for Symmetry Analysis and Mathematical Modelling (ISAMM) organized by N.H. Ibragimov at the University of North-West, South Africa
- **2000** - The research centre for Advances in Lie Group Analysis (ALGA) was organized at Blekinge Institute of Technology, Sweden.

To make results of ALGA widely available, we now publish the first volume of the Archives of ALGA. Our gratitude is due to the rector of Blekinge Institute of Technology, Professor Lars Haikola, for his everlasting support for ALGA.
Förord


Milstolpar

- **1983** - Ett forskningscentrum för Liegruppanalys grundades av N.H. Ibragimov vid Aviation Technical University, Ufa, Ryssland
- **1987** - Forskningscentrat fortsatte sin verksamhet vid Institute of Applied Mathematics och därefter vid Institute of Mathematical Modelling of the USSR Academy of Sciences i Moskva.
- **1994** - Internationell verksamhet etablerades vid Centre for Symmetry Analysis and Differential Equations vid University of the Witwatersrand, Johannesburg, Sydafrika
- **1998** - Verksamheten fortsatte som International Institute for Symmetry Analysis and Mathematical Modelling (ISAMM) organiserat av N.H. Ibragimov vid University of North-West, Sydafrika
- **2000** - Det internationella forskningscentrat Advances in Lie Group Analysis (ALGA) organiserades vid Blekinge Tekniska Högskola, Karlskrona.

För att göra ALGA:s forskningsresultat allmänt tillgängliga publicerar vi härmed den första volymen av Archives of ALGA. Vi vill i samband med detta uttrycka vår tacksamhet för det stöd vi ständigt fått av rektorn för Blekinge Tekniska Högskola, Lars Haikola.

Professor N.H. Ibragimov
22/6 2004
# Contents

**Preface** 3  
**Förord** 4  

*Nail H. Ibragimov*

**Equivalence groups and invariants of linear and non-linear equations** 9  
1 Introduction .................................................. 9  
2 Two methods for calculation of equivalence groups ................. 16  
  2.1 Equivalence transformations for $y'' = F(x,y)$ .................. 16  
  2.2 Infinitesimal method illustrated by equation $y'' = F(x,y)$ .. 18  
  2.3 Equivalence group for linear ordinary differential equations .... 20  
  2.4 A system of linear ordinary differential equations .............. 22  
3 Equivalence group for the filtration equation ..................... 25  
  3.1 Secondary prolongation and the infinitesimal method ............ 26  
  3.2 Direct search for the equivalence group $\mathcal{E}$ ............. 30  
  3.3 Two equations related to the filtration equation ............... 31  
4 Equivalence groups for non-linear wave equations ................ 32  
  4.1 The equations $v_{tt} = f(x,v_x)v_{xx} + g(x,v_x)$ .............. 32  
  4.2 The equations $u_{tt} - u_{xx} = f(u,u_t,u_x)$ .................. 35  
5 Equivalence groups for evolution equations ....................... 35  
  5.1 The generalised Burgers equation ............................ 35  
  5.2 The equations $u_t = f(x,u)u_{xx} + g(x,u,u_x)$ ............... 36  
  5.3 A model from tumour biology .................................. 37  
6 Examples from non-linear acoustics .......................... 37  
7 Invariants of linear ordinary differential equations ............ 39  
8 Invariants of hyperbolic second-order linear partial differential equations in two variables ......................... 42  
  8.1 Equivalence transformations .................................. 42  
  8.2 Semi-invariants ............................................. 43  
  8.3 Laplace’s problem. Calculation of invariants .................. 46
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.4</td>
<td>Invariant differentiation and a basis of invariants. Solution of</td>
<td>51</td>
</tr>
<tr>
<td></td>
<td>Laplace’s problem</td>
<td></td>
</tr>
<tr>
<td>8.5</td>
<td>Representation of invariants in alternative coordinates</td>
<td>54</td>
</tr>
<tr>
<td>9</td>
<td>Invariants of elliptic equations</td>
<td>58</td>
</tr>
<tr>
<td>10</td>
<td>Semi-invariants of parabolic equations</td>
<td>61</td>
</tr>
<tr>
<td>11</td>
<td>Invariants of non-linear wave equations</td>
<td>63</td>
</tr>
<tr>
<td>11.1</td>
<td>The equations $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$</td>
<td>63</td>
</tr>
<tr>
<td>11.2</td>
<td>The equations $u_{tt} - u_{xx} = f(u, u_t, u_x)$</td>
<td>64</td>
</tr>
<tr>
<td>12</td>
<td>Invariants of generalised Burgers equations</td>
<td>65</td>
</tr>
</tbody>
</table>

Nail H. Ibragimov and Sergey V. Meleshko

**Linearization of third-order ordinary differential equations**

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction</td>
<td>71</td>
</tr>
<tr>
<td>1.1</td>
<td>Second-order equations: Lie’s linearization test</td>
<td>72</td>
</tr>
<tr>
<td>1.2</td>
<td>Third-order equations: Candidates for linearization</td>
<td>74</td>
</tr>
<tr>
<td>2</td>
<td>Formulation of the linearization theorems</td>
<td>75</td>
</tr>
<tr>
<td>2.1</td>
<td>The linearization test for Equation (1.18)</td>
<td>76</td>
</tr>
<tr>
<td>2.2</td>
<td>The linearization test for Equation (1.19)</td>
<td>77</td>
</tr>
<tr>
<td>3</td>
<td>Relations between coefficients and transformations</td>
<td>79</td>
</tr>
<tr>
<td>3.1</td>
<td>The coefficients of Equation (1.18)</td>
<td>80</td>
</tr>
<tr>
<td>3.2</td>
<td>The coefficients of Equation (1.19)</td>
<td>80</td>
</tr>
<tr>
<td>4</td>
<td>Proof of the linearization theorems</td>
<td>82</td>
</tr>
<tr>
<td>4.1</td>
<td>Equation (1.18)</td>
<td>82</td>
</tr>
<tr>
<td>4.2</td>
<td>Equation (1.19)</td>
<td>84</td>
</tr>
<tr>
<td>5</td>
<td>Examples to linearization theorems</td>
<td>87</td>
</tr>
<tr>
<td>5.1</td>
<td>Examples on Theorem 2.1</td>
<td>87</td>
</tr>
<tr>
<td>5.2</td>
<td>An example on Theorem 2.2</td>
<td>90</td>
</tr>
</tbody>
</table>

Nail H. Ibragimov, Gazanfer ¨Unal and Claes Jogr´eus

**Group analysis of stochastic differential systems:**

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction</td>
<td>95</td>
</tr>
<tr>
<td>2</td>
<td>Approximate symmetries and conservation laws of deterministic</td>
<td>97</td>
</tr>
<tr>
<td></td>
<td>differential equations</td>
<td></td>
</tr>
<tr>
<td>2.1</td>
<td>Approximate transformations</td>
<td>97</td>
</tr>
<tr>
<td>2.2</td>
<td>Approximate symmetries</td>
<td>100</td>
</tr>
<tr>
<td>2.3</td>
<td>Approximate version of Noether’s theorem</td>
<td>104</td>
</tr>
<tr>
<td>3</td>
<td>Stochastic differential equations</td>
<td>106</td>
</tr>
<tr>
<td>3.1</td>
<td>Stochastic processes</td>
<td>106</td>
</tr>
<tr>
<td>3.2</td>
<td>The Itô and Stratonovich integrals</td>
<td>107</td>
</tr>
<tr>
<td>Section</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>4</td>
<td>Approximate symmetries of Itô dynamical systems</td>
<td>109</td>
</tr>
<tr>
<td>5</td>
<td>Approximate symmetries of Stratonovich dynamical systems</td>
<td>112</td>
</tr>
<tr>
<td>6</td>
<td>The relation between approximate symmetries of the Fokker-Planck equation and the Itô system</td>
<td>113</td>
</tr>
<tr>
<td>7</td>
<td>Approximate conservation laws for stochastic dynamical systems</td>
<td>116</td>
</tr>
<tr>
<td>7.1</td>
<td>Preliminaries on differential forms</td>
<td>116</td>
</tr>
<tr>
<td>7.2</td>
<td>Conservation laws</td>
<td>118</td>
</tr>
<tr>
<td>8</td>
<td>An Application</td>
<td>120</td>
</tr>
</tbody>
</table>
Abstract. Recently I developed a systematic method for determining invariants of families of equations. The method is based on the infinitesimal approach and is applicable to algebraic and differential equations possessing finite or infinite equivalence groups. Moreover, it does not depend on the assumption of linearity of equations. The method was applied to variety of ordinary and partial differential equations. The present paper is aimed at discussing the main principles of the method and its applications with emphasis on the use of infinite Lie groups.

1 Introduction

The concept of invariants of differential equations is commonly in the case of linear second-order ordinary differential equations

\[ y'' + 2c_1(x)y' + c_2(x)y = 0. \]

Namely, the linear substitution (an equivalence transformation)

\[ \tilde{y} = \sigma(x)y \]

maps our equation again in a linear second-order equation and does not change the value of the the quantity

\[ J = c_2 - c_1^2 - c_1'. \]

Knowledge of the invariant is useful in integration of differential equations. For instance, the equation

\[ x^2y'' + xy' + \left( x^2 - \frac{1}{4} \right)y = 0 \]

has the invariant \( J = 1 \). The invariant \( \tilde{J} \) of the equation

\[ \tilde{y}'' + \tilde{y} = 0 \]
has the same value, \( \tilde{J} = 1 \). In consequence, the first equation can be reduced to the second one by an equivalence transformation and hence readily integrated.

Mathematicians came across invariant quantities for families of equations in the very beginning of the theory of partial differential equations. The first partial differential equation, the wave equation \( u_{xy} = 0 \) for vibrating strings, was formulated and solved by d’Alembert in 1747. Two invariant quantities, \( h \) and \( k \), for linear hyperbolic equations were found in 1769/1770 by Euler [1], then in 1773 by Laplace [2]. These fundamental invariant quantities are known today as the Laplace invariants.

We owe to Leonard Euler the first significant results in integration theory of general hyperbolic equations with two independent variables \( x, y \):

\[
\frac{\partial^2 u}{\partial x \partial y} + a(x, y)\frac{\partial u}{\partial x} + b(x, y)\frac{\partial u}{\partial y} + c(x, y)u = 0. \tag{1.1}
\]

In his ”Integral calculus” [1], Euler introduced what is known as the Laplace invariants, \( h \) and \( k \). Namely, he generalized d’Alembert’s solution and showed that Eq. (1.1) is factorable, and hence integrable by solving two first-order ordinary differential equations, if and only if its coefficients \( a, b, c \) obey one of the following equations:

\[
h \equiv a_x + ab - c = 0, \tag{1.2}
\]

or

\[
k \equiv b_y + ab - c = 0. \tag{1.3}
\]

If \( h = 0 \), Eq. (1.1) is factorable in the form

\[
\left( \frac{\partial}{\partial x} + b \right) \left( \frac{\partial u}{\partial y} + au \right) = 0. \tag{1.4}
\]

Then setting

\[
v = u_y + au \tag{1.5}
\]

we rewrite Eq. (1.4) as a first-order equation \( v_x + bv = 0 \) and integrate to obtain:

\[
v = B(y)e^{- \int b(x,y)dx}. \tag{1.6}
\]

Now substitute (1.6) in (1.5), integrate the resulting non-homogeneous linear equation

\[
u_y + au = B(y)e^{- \int b(x,y)dx} \tag{1.7}
\]

with respect to \( y \) and obtain the following general solution to Eq. (1.1):

\[
u = \left[ A(x) + \int B(y)e^{\int ady - bdx}dy \right] e^{- \int ady} \tag{1.8}
\]

with two arbitrary functions \( A(x) \) and \( B(y) \).
Likewise, if \( k = 0 \), Eq. (1.1) is factorable in the form

\[
\left( \frac{\partial}{\partial y} + a \right) \left( \frac{\partial u}{\partial x} + bu \right) = 0.
\] (1.9)

In 1773, Laplace [2] developed a more general method than that of Euler. In Laplace’s method, known also as the cascade method, the quantities \( h, k \) play the central part. Laplace introduced two non-point equivalence transformations. Laplace’s first transformation has the form

\[
v = u_y + au,
\] (1.10)

and the second transformation has the form.

\[
w = u_x + bu.
\] (1.11)

Laplace’s transformations allow one to solve some equations when both Laplace invariants are different from zero. Thus, let us assume that \( h \neq 0, k \neq 0 \) and consider the transformation (1.10). It maps Eq. (1.1) to the equation

\[
v_{xy} + a_1 v_x + b_1 v_y + c_1 v = 0
\] (1.12)

with the following coefficients:

\[
a_1 = a - \frac{\partial \ln |h|}{\partial y}, \quad b_1 = b, \quad c_1 = c + b_y - a_x - b \frac{\partial \ln |h|}{\partial y}.
\] (1.13)

The formulae (1.2) give the following Laplace invariants for Eq. (1.12):

\[
h_1 = 2h - k - \frac{\partial^2 \ln |h|}{\partial x \partial y}, \quad k_1 = h.
\] (1.14)

Likewise, one can utilize the second transformation (1.11) and arrive to a linear equation for \( w \) with the Laplace invariants

\[
h_2 = k, \quad k_2 = 2k - h - \frac{\partial^2 \ln |k|}{\partial x \partial y}.
\] (1.15)

If \( h_1 = 0 \), one can solve Eq. (1.12) using Euler’s method described above. Then it remains to substitute the solution \( v = v(x, y) \) in (1.10) and to integrate the non-homogeneous first-order linear equation (1.10) for \( u \). If \( h_1 \neq 0 \) but \( k_2 = 0 \), we find in a similar way the function \( w = w(x, y) \) and solve the non-homogeneous first-order linear equation (1.11) for \( u \). If \( h_1 \neq 0 \) and \( k_2 \neq 0 \), one can iterate the Laplace transformations by applying the transformations (1.10) and (1.11) to equations for \( v \) and \( w \), etc. This is the essence of the cascade method.
In the 1890s, Darboux discovered the invariance of \( h \) and \( k \) and called them the *Laplace invariants*. He also simplified and improved Laplace’s method, and the method became widely known due to Darboux’s excellent presentation (see [3], Book IV, Chapters 2-9). Since the quantities \( h \) and \( k \) are invariant only under a subgroup of the equivalence group rather than the entire equivalence group, I proposed [4] to call \( h \) and \( k \) the *semi-invariants* in accordance with Cayley’s theory of algebraic invariants [5] (see also [6]). Two proper invariants, \( p \) and \( q \) (see further Section 8) were found only in 1960 by Ovsyannikov[7]. Note, that Laplace’s semi-invariants and Ovsyannikov’s invariants were discovered by accident. The question on existence of other invariants remained open. Thus, the problem arose on determination of all invariants for Eqs. (1.1). I called it *Laplace’s problem*. The problem was solved recently in [8].

Louise Petren, in her PhD thesis [9] defended at Lund University in 1911, extended Laplace’s method and the Laplace invariants to higher-order equations. She also gave a good historical exposition which I used in the present paper, in particular, concerning Euler’s priority in discovering the semi-invariants \( h, k \). Unfortunately, her profound results remain unknown until now.

Semi-invariants for linear ordinary differential equations were intensely discussed in the 1870-1880’s by J. Cockle [10], [11], E. Laguerre [12], [6], J.C. Malet [13], G.H. Halphen [14], R. Harley [15], and A.R. Forsyth [16]. The restriction to linear equations was essential in their approach. They used calculations following directly from the definition of invariants. These calculations would be extremely lengthy in the case of non-linear equations. Indeed, when Roger Liouville [17], [18] investigated the invariants for the following class of non-linear ordinary differential equations:

\[
y'' + F_3(x, y)y'^3 + F_2(x, y)y'^2 + F_1(x, y)y' + F(x, y) = 0,
\]

introduced by Lie [19], the direct method led to 70 pages of calculations (cf. [20]).

In his review paper [21], Chapter I, §1.11, Lie noticed: “I refer to the remarkable works of Laguerre (1879) and Halphen (1882) on transformations of ordinary linear differential equations. These investigations in fact deal with the infinite group of transformations \( x' = f(x), y' = g(x)y \) which is mentioned by neither of the authors. I think that Laguerre and Halphen did not know my theory.” Lie himself did not have time to develop this idea. Lie’s remark provided me with an incentive to begin in 1996 the systematic development of a new approach to calculation of invariants by using the infinitesimal technique, that was lacking in old methods. The method was developed in [4] (see also [22], Chapter 10) and subsequently applied to numerous linear and non-linear equations. These applications showed that the method is effective for determining invariants for equations with finite or infinite equivalence groups.

The present paper is a practical guide for calculation of invariants for families of linear and non-linear differential equations with special emphasis on the use of infinite equivalence Lie algebras.
Gallery of main figures and landmarks

[d’Alembert, Jean Le Rond (1717-1783)]
Philosoper/Mathematician

Known for d’Alembert’s principle, Partial differential equations, “Dictionnaire Encyclopédique”.

1747 - d’Alembert’s pioneering work in partial differential equations: he published the renowned wave equation for vibrating strings and solved it (d’Alembert’s solution).

[Euler, Leonard (1707-1783)]

Contributed to many branches of mathematics: algebra, geometry, analysis, variation calculus, differential equations, optics, hydrodynamics and astronomy.

1769 - Euler was the first to introduce what is known as the Laplace invariants. He used them to understand the nature of d’Alembert’s solution and to factorize the general hyperbolic equations. These results were published in [1].

[Laplace, Pierre-Simon (1749-1827)]
Mathematician/Astronomer

Contributed to many branches of mathematics.
In 5 vol-s of his ”Celestial mechanics” (1799-1825)
Laplace systematized the work of 3 generations of illustrious mathematicians.

1773 - Laplace presented a new method, known as Laplace’s “cascade method”, in his fundamental paper “Studies on integral calculus of partial differences” [2]. The central part in his method is played by the semi-invariants $h$ and $k$ known after him as the Laplace invariants. His method allows one to solve many hyperbolic equations.
Laguerre,
Edmond (1834-1886)
Known in analysis for Laguerre’s polynomials.

1879 - Laguerre discovered semi-invariants for the third-order linear ordinary differential equations.

Darboux,
Gaston (1838-1922)
Mathematician, teacher and organizer of science

1890 - In his ”Theory of surfaces” Darboux gave a profound treatment of Laplace’s method and disseminated the Laplace invariants.

Lie,
Sophus (1842-1899)
Created a new branch in mathematics - Lie groups, Lie algebras and group analysis of differential equations.

1895 - Lie [21] made a significant remark on importance of infinite groups in the theory of invariants of differential equations.
**Petrénn, Louise (1880-1977)**

Mathematician

Made a significant contribution to the theory of invariants of partial differential equations.

_The photo is published by courtesy of grandson of L. Petrénn, Professor Lars Haikola._

1911 - Extension of Laplace’s method and Laplace’s invariants to higher-order equations by Louise Pétren in her PhD thesis ”Extension de la méthode de Laplace” [9]. The work contains a good historical introduction starting from Euler’s work. Petrénn’s invariants should be investigated from group point of view.

**Ovsyannikov, Lev (born 1919)**


1960 - Ovsyannikov discovered a new invariant, $q$, and used the invariants $p$ and $q$ in the problem of group classification of hyperbolic equations.

**Ibragimov**

Nail H.

Applied Lie groups in initial value problems and mathematical modelling, developed new methods in group analysis.

Two methods for calculation of equivalence groups

Equivalence transformations play the central part in the theory of invariants discussed in the present paper. The set of all equivalence transformations of a given family of equations forms a group called the equivalence group and denoted by $\mathcal{E}$. The continuous group of equivalence transformations is a subgroup of $\mathcal{E}$ and is denoted by $\mathcal{E}_c$.

In this section, we discuss the notation and illustrate two main methods for calculation of equivalence transformations for families of equations. The first method consists in the direct search for the equivalence transformations and, theoretically, allows one to calculate the most general equivalence group $\mathcal{E}$. The direct method was used by Lie [23] (see also [24]) for calculation of the equivalence transformations and group classification of a family of second-order ordinary differential equations. Lie’s result is discussed in Section 2.1. The direct method is further discussed in Section 3.2 for the nonlinear filtration equations.

However, the direct method leads, in general, to considerable computational difficulties. One will have the similar situation if one will calculate symmetry groups by using Lie’s infinitesimal method and by the direct method.

Therefore, I employ mostly the second method suggested by Ovsyannikov [25] for determining generators of continuous equivalence groups $\mathcal{E}_c$. The central part in this method is played by what I call here a secondary prolongation. This concept leads to a modification of Lie’s infinitesimal method. After simple introductory examples given in Section 2.2 and Section 2.4, I describe in detail the essence of the method in Section 3.1 and Section 4.1.

In what follows, the Lie algebra of the continuous equivalence group $\mathcal{E}_c$ is called the equivalence algebra and is denoted by $L_{\mathcal{E}}$.

2.1 Equivalence transformations for $y'' = F(x, y)$

Following Lie (see [23], §2, p. 440-446), I discuss here the equivalence transformations for the following family of second-order ordinary differential equations:

$$y'' = F(x, y).$$

Definition 2.1. An equivalence transformation of the family of the equations (2.1) is a change of variables

$$\bar{x} = \varphi(x, y), \quad \bar{y} = \psi(x, y)$$

(2.2)

carrying every equation of the form (2.1) into an equation of the same form:

$$\bar{y}'' = \bar{F}(\bar{x}, \bar{y}).$$

(2.3)

The function $\bar{F}$ may be, in general, different from the original function $F$. The equations (2.1) and (2.3) are said to be equivalent.
In this simple example, one can readily find the equivalence transformations by the direct method. Namely, the change of variables (2.2) implies the equations

\[ \bar{y}' \equiv \frac{d\bar{y}}{d\bar{x}} = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y} \]  

(2.4)

and

\[ \bar{y}'' = \frac{\varphi_x + y'\varphi_y \begin{vmatrix} \varphi_{xx} + 2y'\varphi_{xy} + y'^2\varphi_{yy} + y''\varphi_y \\ \psi_x + y'\psi_y \begin{vmatrix} \varphi_{xx} + 2y'\varphi_{xy} + y'^2\varphi_{yy} + y''\psi_y \\ \varphi \cdot (\varphi + y'\varphi_y)^3 \end{vmatrix} } {\begin{vmatrix} \varphi_{xx} + 2y'\varphi_{xy} + y'^2\varphi_{yy} + y''\varphi_y \\ \psi_x + y'\psi_y \begin{vmatrix} \varphi_{xx} + 2y'\varphi_{xy} + y'^2\psi_{yy} + y''\psi_y \\ \varphi \cdot (\varphi + y'\varphi_y)^3 \end{vmatrix} } . \]  

(2.5)

for the change of the first and second derivatives, respectively (see [23], p. 440, or [22], Section 12.3.1). Now we substitute (2.5) in Eq. (2.3) and have:

\[ (\varphi_x + y'\varphi_y)^3F(\bar{x}, \bar{y}) = \begin{vmatrix} \varphi_x + y'\varphi_y \varphi_{xx} + 2y'\varphi_{xy} + y'^2\varphi_{yy} + F(x, y)\varphi_y \\ \psi_x + y'\psi_y \varphi_{xx} + 2y'\psi_{xy} + y'^2\psi_{yy} + F(x, y)\psi_y \end{vmatrix} . \]  

(2.6)

Since \( F(x, y) \) and \( F(\bar{x}, \bar{y}) \) do not depend on \( y' \), Eq. (2.6) splits into four equations obtained by equating to zero the coefficients for different powers of \( y' \).

Collecting in (2.6) the coefficients for \( y'^3 \), \( y'^2 \), \( y' \) and taking into account that \( F(x, y) \), and hence \( F(\bar{x}, \bar{y}) \) are arbitrary functions, we get:

\[ \varphi_y = 0, \quad \varphi_x \psi_{yy} = 0, \quad 2\varphi_x \psi_{xy} - \psi_y \varphi_{xx} = 0. \]

The first equation yields

\[ \varphi = \varphi(x), \]

where \( \varphi(x) \) is and arbitrary function obeying the non-degeneracy condition \( \varphi'(x) \neq 0 \).

The latter condition reduces the second equation \( \psi_{yy} = 0 \), and hence

\[ \psi = \alpha(x)y + \beta(x), \quad \alpha(x) \neq 0. \]

Finally, the third equation becomes

\[ \frac{a'}{a} = \frac{\varphi''}{\varphi'}, \]

whence

\[ a(x) = A \sqrt{|\varphi'(x)|}, \quad A = \text{const}. \]

The remaining term in equation (2.6) does not contain \( y' \) and provides the following expression for the right-hand side \( \bar{F} \) of Eq. (2.3):

\[ \bar{F} = \frac{A}{(\varphi')^{3/2}} F + A \left[ \frac{\varphi''}{2(\varphi')^{5/2}} - \frac{3(\varphi'')^2}{4(\varphi')^{7/2}} \right] y + \frac{\beta''}{(\varphi')^{3}} - \frac{\beta'\varphi''}{(\varphi')^{3}}. \]

Collecting together the above expressions for \( \varphi, \psi \) and \( \bar{F} \), we formulate the result.
Theorem 2.1. The equivalence group $E$ for the equations (2.1) is an infinite group given by the transformations

$$\bar{x} = \varphi(x), \quad \bar{y} = A\sqrt{\varphi'(x)} y + \beta(x), \quad (2.7)$$

$$\overline{F} = \frac{A}{(\varphi')^{3/2}} F + A\left[\frac{\varphi''}{2(\varphi')^{5/2}} - \frac{3(\varphi'')^2}{4(\varphi')^{7/2}}\right] y + \frac{\beta''}{(\varphi')^2} - \frac{\beta' \varphi''}{(\varphi')^3}, \quad (2.8)$$

where $\varphi(x)$ is an arbitrary function such that $\varphi'(x) = 0$, and $A \neq 0$ is an arbitrary constant. In order to obtain the function $\overline{F}(\bar{x}, \bar{y})$, it suffices to express $x, y$ via $\bar{x}, \bar{y}$ from the equations (2.7) and substitute in (2.8).

### 2.2 Infinitesimal method illustrated by equation $y'' = F(x, y)$

Let us find the continuous group $E_c$ of equivalence transformations by means of the infinitesimal method. Since the right-hand side of Eq. (2.1) may change under equivalence transformations, we treat $F$ as a new variable and, adding to (2.2) an arbitrary transformation of $F$, consider the extended transformation:

$$\bar{x} = \varphi(x, y), \quad \bar{y} = \psi(x, y), \quad \overline{F} = \Phi(x, y, F). \quad (2.9)$$

Consequently, we look for the generator of the continuous equivalence group written in the extended space of variables $(x, y, F)$ as follows:

$$Y = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} + \mu(x, y, F) \frac{\partial}{\partial F}. \quad (2.10)$$

Here $y$ is a differential function with one independent variable $x$, whereas $F$ is a differential function with two independent variables $x, y$. Accordingly, the prolongation of $Y$ to $y''$ is given by the usual prolongation procedure, namely:

$$\tilde{Y} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \mu \frac{\partial}{\partial F} + \zeta_2 \frac{\partial}{\partial y''},$$

where

$$\zeta_2 = \eta_{xx} + (2\eta_{xy} - \xi_{xx}) y' + (\eta_{yy} - 2\xi_{xy}) y'^2 - \xi_{yy} y'^3 + (\eta_y - 2\xi_x - 3\xi_y y') y''.$$

The infinitesimal invariance test of Eq. (2.1) has the form

$$\zeta_2 \bigg|_{y''=F} = \mu.$$ 

Substituting here the expression for $\zeta_2$, we have:

$$\eta_{xx} + (2\eta_{xy} - \xi_{xx}) y' + (\eta_{yy} - 2\xi_{xy}) y'^2 - \xi_{yy} y'^3 + (\eta_y - 2\xi_x - 3\xi_y y') F = \mu(x, y, F), \quad (2.10)$$
where \( F \) is a variable, not a function \( F(x, y) \). Hence, Eq. (2.10) should be satisfied identically in the independent variables \( x, y, y', \) and \( F \). Accordingly, we split Eq. (2.10) into four equations by annulling the terms with different powers of \( y' \). Since \( \mu(x, y, F) \) does not depend on \( y' \), we obtain the following equations:

\[
\begin{align*}
y'^3 : & \quad \xi_{yy} = 0, \\
y'^2 : & \quad \eta_{yy} - 2\xi_{xy} = 0, \\
y' : & \quad 2\eta_{xy} - \xi_{xx} - 3\xi_y F = 0, \\
y'^0 : & \quad \mu = (\eta_y - 2\xi_x)F + \eta_{xx}.
\end{align*}
\]

Invoking that \( \xi \) and \( \eta \) do not depend upon \( F \), we split Eq. (2.13) into two equations:

\[
\begin{align*}
x_y = 0, & \quad 2\eta_{xy} - \xi_{xx} = 0.
\end{align*}
\]

The first equation yields \( \xi = \xi(x) \). Then the second equation is written \( 2\eta_{xy} = \xi'(x) \), whence upon integration:

\[
\eta = \left[ \frac{1}{2} \xi'(x) + C \right] y + \beta(x).
\]

Now the equations (2.11)-(2.13) are manifestly satisfied, and the remaining equation (2.14) yields:

\[
\mu = \left[ C - \frac{3}{2} \xi'(x) \right] F + \frac{1}{2} \xi''(x)y + \beta''(x), \quad C = \text{const}.
\]

Thus, the general solution of equations (2.11)-(2.14) has the form

\[
\begin{align*}
\xi & = \xi(x), \quad \eta = \left[ \frac{1}{2} \xi'(x) + C \right] y + \beta(x), \\
\mu & = \left[ C - \frac{3}{2} \xi'(x) \right] F + \frac{1}{2} \xi''(x)y + \beta''(x).
\end{align*}
\]

We summarize.

**Theorem 2.2.** The continuous equivalence group \( E_c \) for Eqs. (2.1) is an infinite group. The corresponding equivalence algebra \( L_E \) is spanned by the operators

\[
\begin{align*}
Y_0 &= y \frac{\partial}{\partial y} + F \frac{\partial}{\partial F}, \\
Y_\beta &= \beta(x) \frac{\partial}{\partial y} + \beta''(x) \frac{\partial}{\partial F}, \\
Y_\xi &= \xi(x) \frac{\partial}{\partial x} + \frac{y}{2} \xi'(x) \frac{\partial}{\partial y} + \left[ \frac{y}{2} \xi''(x) - \frac{3}{2} \xi'(x)F \right] \frac{\partial}{\partial F}.
\end{align*}
\]

**Remark 2.1.** Equations (2.14) can be obtained from Theorem 2.1 by letting \( A > 0 \), setting \( \varphi(x) = x + a\xi(x) \) with a small parameter \( a \), and writing Eqs. (2.7)-(2.8) in the first order of precision with respect to \( a \). Hence, the transformations of the continuous equivalence group \( E_c \) have the form (2.7)-(2.8), where \( A > 0 \). Thus, the continuous equivalence group \( E_c \) differs from the general equivalence group \( E \) given by Theorem 2.1 only by the restriction \( A > 0 \).
2.3 Equivalence group for linear ordinary differential equations

In theory of invariants, it is advantageous to write linear homogeneous ordinary differential equations of the \( n \)th order in a *standard form* involving the binomial coefficients:

\[
L_n(y) \equiv y^{(n)} + nc_1 y^{(n-1)} + \frac{n(n-1)}{2!}c_2 y^{(n-2)} + \cdots + nc_{n-1} y' + c_n y = 0, \quad (2.17)
\]

where \( c_i = c_i(x) \) are arbitrary variable coefficients, and \( y' = dy/dx \), etc.

An equivalence transformation of the equations (2.17) is an invertible transformation (2.2) of the independent variable \( x \) and the dependent variables \( y \) preserving the order \( n \) of any equation (2.17) and its linearity and homogeneity. Recall the well-known classical result.

**Theorem 2.3.** The set of all equivalence transformations of the equations (2.17) is an infinite group composed of the linear transformation of the dependent variable:

\[
\bar{x} = x, \quad \bar{y} = \phi(x)y,
\]

(2.18)

where \( \phi(x) \neq 0 \), and an arbitrary change of the independent variable:

\[
\bar{x} = f(x), \quad \bar{y} = y,
\]

(2.19)

where \( f'(x) \neq 0 \).

In calculation of invariants of linear equations, we will use in Section 7 the infinitesimal form of the extension (cf. (2.9)) of each transformation (2.18) and (2.19) to the coefficients of Eq. (2.17).

Let us find the extension of the infinitesimal transformation (2.18) for the equation (2.17) of the second order,

\[
L_2(y) \equiv y'' + 2c_1(x)y' + c_2(x)y = 0,
\]

(2.20)

and for the equation of the third order,

\[
L_3(y) \equiv y''' + 3c_1(x)y'' + 3c_2(x)y' + c_3(x)y = 0. \quad (2.21)
\]

We implement the infinitesimal transformation (2.18) by letting

\[
\phi(x) = 1 - \varepsilon \eta(x)
\]

(2.22)

with a small parameter \( \varepsilon \). Then

\[
y \approx (1 - \varepsilon \eta) \bar{y},
y' \approx (1 - \varepsilon \eta) \bar{y}' - \varepsilon \eta' \bar{y},
y'' \approx (1 - \varepsilon \eta) \bar{y}'' - \varepsilon(2\eta' \bar{y}' + \eta'' \bar{y}),
y''' \approx (1 - \varepsilon \eta) \bar{y}''' - \varepsilon(3\eta' \bar{y}'' + 3\eta'' \bar{y}' + \eta''' \bar{y}).
\]
Substituting these expressions in Eq. (2.20), dividing by \((1 - \varepsilon\eta)\) and noting that \(\varepsilon/(1 - \varepsilon\eta) \approx \varepsilon\), one obtains:

\[
L_2(y) \approx \ddot{y}'' + 2[c_1 - \varepsilon\eta'] \dot{y}' + [c_2 - \varepsilon(\eta'' + 2c_1\eta')]\dot{y}.
\]

Hence, the infinitesimal equivalence transformation (2.18) maps Eq. (2.20) into an equivalent equation:

\[
\ddot{y}'' + 2\overline{c}_1(x) \dot{y}' + \overline{c}_2(x) \ddot{y} = 0,
\]

where

\[
\overline{c}_1 \approx c_1 - \varepsilon\eta', \quad \overline{c}_2 \approx c_2 - \varepsilon(\eta'' + 2c_1\eta').
\]  

(2.24)

Eqs. (2.24), together with the equation \(y \approx (1 - \varepsilon\eta) \bar{y}\), provide the following generator of the equivalence transformation (2.18) extended to the coefficients of the second-order equation (2.20):

\[
Y_\eta = \eta \frac{\partial}{\partial y} + \eta' \frac{\partial}{\partial c_1} + (\eta'' + 2c_1\eta') \frac{\partial}{\partial c_2}.
\]  

(2.25)

Likewise, the third-order equation (2.21) is transformed into an equivalent equation

\[
\ddot{y}''' + 3\overline{c}_1(x) \ddot{y}'' + 3\overline{c}_2(x) \dot{y}' + \overline{c}_3(x) \ddot{y} = 0,
\]

where

\[
\overline{c}_1 \approx c_1 - \varepsilon\eta', \quad \overline{c}_2 \approx c_2 - \varepsilon(\eta'' + 2c_1\eta'), \quad \overline{c}_3 \approx c_3 - \varepsilon(\eta''' + 3c_1\eta'' + 3c_2\eta').
\]  

(2.27)

Eqs. (2.27), together with the equation \(y \approx (1 - \varepsilon\eta) \bar{y}\), provide the following generator of the equivalence transformation (2.18) extended to the coefficients of the third-order equation (2.21):

\[
Y_\eta = \eta \frac{\partial}{\partial y} + \eta' \frac{\partial}{\partial c_1} + (\eta'' + 2c_1\eta') \frac{\partial}{\partial c_2} + (\eta''' + 3c_1\eta'' + 3c_2\eta') \frac{\partial}{\partial c_3}.
\]  

(2.28)

Let us find the extension of the infinitesimal transformation (2.19) for the third-order equation (2.21). We take the infinitesimal transformation (2.21),

\[
\bar{x} \approx x + \varepsilon\xi(x)
\]

and have:

\[
\begin{align*}
y' & \approx (1 + \varepsilon\xi')\dot{y}', \\
y'' & \approx (1 + 2\varepsilon\xi')\ddot{y}'' + \varepsilon\dot{y}'\xi'', \\
y''' & \approx (1 + 3\varepsilon\xi')\dddot{y}''' + 3\varepsilon\ddot{y}''\xi'' + \varepsilon\dot{y}''\xi'''.
\end{align*}
\]
Consequently Eq. (2.21) becomes
\[ \dddot{y} + 3c_1 \ddot{y} + 3c_2 \dot{y} + c_3 y = 0, \]
where
\[ c_1 \approx c_1 + \varepsilon (\xi' - c_1 \xi), \quad c_2 \approx c_2 + \varepsilon \left( \frac{1}{3} \xi''' + c_1 \xi'' - 2c_2 \xi' \right), \quad c_3 \approx c_3 - 3c_3 \xi'. \]

The corresponding group generator is
\[ X_\xi = \xi \frac{\partial}{\partial x} + \left( \xi'' - c_1 \xi' \right) \frac{\partial}{\partial c_1} + \left( \frac{1}{3} \xi''' + c_1 \xi'' - 2c_2 \xi' \right) \frac{\partial}{\partial c_2} - 3c_3 \xi' \frac{\partial}{\partial c_3}. \] (2.29)

### 2.4 A system of linear ordinary differential equations

Let us calculate the continuous equivalence group \( \mathcal{E}_c \) for the following system of linear second-order ordinary differential equations:
\[ x'' + V(t) x = 0, \quad y'' - V(t) y = 0. \] (2.30)

An equivalence transformation of the system (2.30) is a change of variables \( t, x, y \) :
\[ \bar{t} = \alpha(t, x, y, a), \quad \bar{x} = \beta(t, x, y, a), \quad \bar{y} = \gamma(t, x, y, a) \] (2.31)
mapping the system (2.30) into a system of the same form,
\[ \frac{d^2 \bar{x}}{d\bar{t}^2} + V(\bar{t}) \bar{x} = 0, \quad \frac{d^2 \bar{y}}{d\bar{t}^2} - V(\bar{t}) \bar{y} = 0, \]
where the function \( V(\bar{t}) \) can, in general, be different from the original function \( V(t) \). Accordingly, we write the equivalence transformation (2.32) and the system (2.30) in the following extended forms:
\[ \bar{t} = \alpha(t, x, y), \quad \bar{x} = \beta(t, x, y), \quad \bar{y} = \gamma(t, x, y), \quad \bar{V} = \Psi(t, x, y, V), \] (2.32)
and
\[ x'' + x V = 0, \quad y'' - y V = 0, \quad V_x = 0, \quad V_y = 0, \] (2.33)
respectively. Here, \( x \) and \( y \) are, as before, the differential variables with the independent variable \( t \), whereas \( V \) is a new differential variable with three independent variables \( t, x \) and \( y \). Consequently, the infinitesimal generator of a one-parameter group of equivalence transformations is written in the form
\[ Y = \tau(t, x, y) \frac{\partial}{\partial t} + \xi(t, x, y) \frac{\partial}{\partial x} + \eta(t, x, y) \frac{\partial}{\partial y} + \mu(t, x, y, V) \frac{\partial}{\partial V}. \] (2.34)
The extension of the operator (2.34) to all quantities involved in Eqs. (2.33) has the form (see [25], [26] and [27]):

$$\tilde{Y} = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \mu \frac{\partial}{\partial V} + \xi_1 \frac{\partial}{\partial x'} + \zeta_2 \frac{\partial}{\partial y'} + \omega_1 \frac{\partial}{\partial V_x} + \omega_2 \frac{\partial}{\partial V_y}.$$  (2.35)

The condition that \(Y\) is a generator of an equivalence group is equivalent to the statement that \(\tilde{Y}\) satisfies the infinitesimal invariance test for the extended system (2.33). This gives the following determining equations:

$$\zeta_1 \bigg|_{(2.33)} + \xi V + x \mu = 0, \quad \zeta_2 \bigg|_{(2.33)} - \eta V - y \mu = 0,$$  (2.36)

$$\omega_1 \bigg|_{(2.33)} = 0, \quad \omega_2 \bigg|_{(2.33)} = 0,$$  (2.37)

where \(\zeta_1, \zeta_2\) are given by the usual prolongation procedure. Namely,

$$\zeta_1 = D_t(\xi) - x' D_t(\tau) = \xi_t + x' \xi_x + y' \xi_y - x'(\tau_t + x' \tau_x + y' \tau_y),$$

$$\zeta_2 = D_t(\eta) - y' D_t(\tau) = \eta_t + x' \eta_x + y' \eta_y - y'(\tau_t + x' \tau_x + y' \tau_y).$$  (2.39)

The coefficients \(\omega_1\) and \(\omega_2\) are determined by

$$\omega_1 = \tilde{D}_x(\mu) - V_x \tilde{D}_x(\xi) - V_y \tilde{D}_x(\eta) - V_t \tilde{D}_x(\tau), \quad \omega_2 = \tilde{D}_y(\mu) - V_x \tilde{D}_y(\xi) - V_y \tilde{D}_y(\eta) - V_t \tilde{D}_y(\tau).$$  (2.40)

We used here the notation

$$D_t = \frac{\partial}{\partial t} + x' \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + x'' \frac{\partial}{\partial x'} + y'' \frac{\partial}{\partial y'},$$

for the usual total differentiation with respect to \(t\), while

$$\tilde{D}_x = \frac{\partial}{\partial x} + V_x \frac{\partial}{\partial V}, \quad \tilde{D}_y = \frac{\partial}{\partial y} + V_y \frac{\partial}{\partial V}.$$  (2.41)

denote the “new” total differentiations for the extended system (2.33).

The restriction to Eqs. (2.33) means, in particular, that we set \(V_x = V_y = 0\). Then the expressions (2.41) and (2.40) take the form:

$$\tilde{D}_x = \frac{\partial}{\partial x}, \quad \omega_1 = \mu - V_t \tau_x,$$

$$\tilde{D}_y = \frac{\partial}{\partial y}, \quad \omega_2 = \mu - V_t \tau_y.$$
Let us solve the determining equations. We begin with the equations (2.37):
\[
\omega_1 \equiv \mu_x - V_t \tau_x = 0, \quad \omega_2 \equiv \mu_y - V_t \tau_y = 0.
\]
Since \(V\) and hence \(V_t\) are arbitrary functions, the above equations yield:
\[
\tau_x = 0, \quad \mu_x = 0, \quad \tau_y = 0, \quad \mu_y = 0.
\]
Thus, the operator (2.34) reduces to the form
\[
Y = \frac{\partial}{\partial t} + \xi(t, x, y) \frac{\partial}{\partial x} + \eta(t, x, y) \frac{\partial}{\partial y} + \mu(t, V) \frac{\partial}{\partial V} . \tag{2.42}
\]
Let us turn now to the remaining determining equations (2.36). We apply to the operator (2.42) the prolongation formulae (2.38) and substitute the resulting expression for \(\zeta_1\) in the first equation (2.36) to obtain:
\[
\xi_{tt} + (2 \xi_{tx} - \tau'')x' + 2 \xi_{ty} y' + \xi_{xx} x'^2 + 2 \xi_{xy} x' y'
+ \xi_{yy} y'^2 + (y \xi_y - x \xi_x + 2 x \tau' + \xi)V + x \mu = 0. \tag{2.43}
\]
We collect here the like terms and annul the coefficients of different powers of \(x'\) and \(y'\). The coefficients for \(x'^2, x'y', y'^2\), and \(y'\) yield:
\[
\xi_{xx} = 0, \quad \xi_{xy} = 0, \quad \xi_{yy} = 0, \quad \xi_{ty} = 0,
\]
whence
\[
\xi = a(t)x + Ay + k(t), \quad A = \text{const}.
\]
Furthermore, annulling the coefficient \(2 \xi_{tx} - \tau'' = 0\) for \(x'\) we have \(2a'(t) = \tau''(t)\), and hence
\[
a(t) = \frac{1}{2} \tau'(t) + C_1.
\]
Thus,
\[
\xi = \left( \frac{1}{2} \tau'(t) + K_1 \right) x + Ay + k(t). \tag{2.44}
\]
Now Eq. (2.43) becomes
\[
\frac{x}{2} \tau'''(t) + k''(t) + [2Ay + k(t) + 2x \tau'(t)]V + x \mu = 0. \tag{2.45}
\]
Likewise, the second equation (2.36) yields
\[
\eta = \left( \frac{1}{2} \tau'(t) + K_2 \right) y + Bx + l(t) \tag{2.46}
\]
and
\[
\frac{y}{2} \tau'''(t) + l''(t) - [2Bx + l(t) + 2y \tau'(t)]V - y \mu = 0. \tag{2.47}
\]
Since \( V \) is regarded as an arbitrary variable, and \( \mu \) does not depend upon \( x \) and \( y \), Eq. (2.45) yields \( A = 0, \ k(t) = 0 \) and
\[
\mu = -\frac{1}{2} \tau'''(t) - 2\tau'(t)V. \tag{2.48}
\]
Likewise, Eq. (2.47) yields \( B = 0, \ l(t) = 0 \) and
\[
\mu = \frac{1}{2} \tau'''(t) - 2\tau'(t)V. \tag{2.49}
\]
Eqs. (2.48) and (2.49) yield that \( \tau'''(t) = 0, \ \mu = -2\tau'(t)V. \) Summing up, we obtain :
\[
\tau(t) = C_3 + C_4 t + C_5 t^2, \nonumber
\]
\[
\xi = (C_1 + C_5 t)x, \nonumber
\]
\[
\eta = (C_2 + C_5 t)y, \nonumber
\]
\[
\mu = -2(C_4 + 2C_5 t)V. \tag{2.50}
\]
We summarize.

**Theorem 2.4.** The equivalence algebra \( L_E \) for the system (2.30) is a five-dimensional Lie algebra spanned by
\[
Y_1 = x \frac{\partial}{\partial x}, \quad Y_2 = y \frac{\partial}{\partial y}, \quad Y_3 = t \frac{\partial}{\partial t} - 2V \frac{\partial}{\partial V},
\]
\[
Y_4 = \frac{\partial}{\partial t}, \quad Y_5 = t^2 \frac{\partial}{\partial t} + tx \frac{\partial}{\partial x} + ty \frac{\partial}{\partial y} - 4tV \frac{\partial}{\partial V}. \tag{2.51}
\]
Note that the operator \( Y_5 \) generates the one-parameter group of transformations
\[
\bar{t} = \frac{t}{1 - at}, \quad \bar{x} = \frac{x}{1 - at}, \quad \bar{y} = \frac{y}{1 - at}, \quad \bar{V} = (1 - at)^4 V. \tag{2.52}
\]

3 Equivalence group for the filtration equation

In this section, we discuss both methods for calculation of equivalence transformations for partial differential equations by considering the nonlinear filtration equation
\[
v_t = h(v_x)v_{xx}. \tag{3.1}
\]
Equation (3.1) is used in mechanics as a mathematical model in studying shear currents of nonlinear viscoplastic media, processes of filtration of non-Newtonian fluids, as well as for describing the propagation of oscillations of temperature and salinity to depths in oceans (see, e.g. [26], Chapter 2, and the references therein). The function \( h(v_x) \) is known as a filtration coefficient. In general, the filtration coefficient is not fixed, and we consider the family of equations of the form (3.1) with an arbitrary function \( h(v_x) \).
**Definition 3.1.** An equivalence transformation of the family of equations of the form (3.1) is a changes of variables
\[
\bar{x} = \varphi(t, x, v), \quad \bar{t} = \psi(t, x, v), \quad \bar{v} = \Phi(t, x, v) \tag{3.2}
\]
carrying every equation of the form (3.1) with any filtration coefficient \(h(v_x)\) into an equation of the same form:
\[
\bar{v}_t = \bar{h}(\bar{v}_x)\bar{v}_{xx}. \tag{3.3}
\]
The function \(\bar{h}\) representing a new filtration coefficient \(\bar{h}(\bar{v}_x)\) may be, in general, different from the original function \(h\).

### 3.1 Secondary prolongation and the infinitesimal method

Equation (3.1) provides a good example for introducing the concept of a secondary prolongation and illustrating the infinitesimal method for calculating the continuous equivalence group \(E_c\).

In order to find the continuous group \(E_c\) of equivalence transformations (3.2), we search for the generators of the group \(E_c\):
\[
Y = \xi_1 \frac{\partial}{\partial t} + \xi_2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial v} + \mu \frac{\partial}{\partial h}. \tag{3.4}
\]
The generator \(Y\) defines the group \(E_c\) of equivalence transformations
\[
\bar{x} = \varphi(t, x, v), \quad \bar{t} = \psi(t, x, v), \quad \bar{v} = \Phi(t, x, v), \quad \bar{h} = F(t, x, v, v_t, v_x, h) \tag{3.5}
\]
for the family of equations (3.1) if and only if \(Y\) obeys the condition of invariance of the following system:
\[
v_t = hv_{xx}, \quad h_t = 0, \quad h_x = 0, \quad h_v = 0, \quad h_{vv} = 0. \tag{3.6}
\]
In order to write the infinitesimal invariance test for the system (3.6), we should extend the action of the operator (3.4) to all variables involved in (3.6), i.e. take
\[
\bar{Y} = Y + \zeta_1 \frac{\partial}{\partial v_t} + \zeta_2 \frac{\partial}{\partial v_x} + \zeta_{22} \frac{\partial}{\partial v_{xx}} + \omega_1 \frac{\partial}{\partial h_t} + \omega_2 \frac{\partial}{\partial h_x} + \omega_3 \frac{\partial}{\partial h_v} + \omega_4 \frac{\partial}{\partial h_{vv}}.
\]
First, we extend \(Y\) to the derivatives of \(v_t, v_x\) and \(v_{xx}\) by treating \(v\) as a differential variable depending on \((t, x)\), as we do in Eq. (3.1). The unknown coordinates \(\xi_1, \xi_2\) and \(\eta\) of the operator (3.4) are sought as functions of the variables \(t, x, v\). Accordingly, we use the usual total differentiations in the space \((t, x, v)\):
\[
D_1 \equiv D_t = \frac{\partial}{\partial t} + v_t \frac{\partial}{\partial v} + v_{tt} \frac{\partial}{\partial v_t} + v_{tx} \frac{\partial}{\partial v_x},
\]
\[
D_2 \equiv D_x = \frac{\partial}{\partial x} + v_x \frac{\partial}{\partial v} + v_{tx} \frac{\partial}{\partial v_t} + v_{xx} \frac{\partial}{\partial v_x}.
\]
and calculate the coordinates \( \zeta_1, \zeta_2 \) and \( \zeta_{22} \) by the usual prolongation formulae:

\[
\zeta_i = D_i(\eta) - v_tD_i(\xi^1) - v_xD_i(\xi^2), \quad (3.7)
\]

\[
\zeta_{22} = D_2(\zeta_2) - v_{tx}D_2(\xi^1) - v_{xx}D_2(\xi^2). \quad (3.8)
\]

Then we pass to the extended space \( (t, x, v, v_t, v_x) \) and consider \( h \) as a differential variable depending on the independent variables \( (t, x, v, v_t, v_x) \). The crucial step of the secondary prolongation is that we consider the coordinate \( \mu \) of the equivalence operator \( (3.4) \) as a function of \( t, x, v, v_t, v_x, h \) and introduce the new total differentiations in the extended space \( t, x, v, v_t, v_x, h \):

\[
\tilde{D}_1 \equiv \tilde{D}_t = \frac{\partial}{\partial t} + h_t \frac{\partial}{\partial h} + h_{tt} \frac{\partial}{\partial h_t} + h_{tx} \frac{\partial}{\partial h_x} + h_{tv} \frac{\partial}{\partial h_v} + h_{tv} \frac{\partial}{\partial h_{vt}},
\]

\[
\tilde{D}_2 \equiv \tilde{D}_x = \frac{\partial}{\partial x} + h_x \frac{\partial}{\partial h} + h_{xt} \frac{\partial}{\partial h_t} + h_{xx} \frac{\partial}{\partial h_x} + h_{xv} \frac{\partial}{\partial h_v} + h_{xv} \frac{\partial}{\partial h_{vt}},
\]

\[
\tilde{D}_3 \equiv \tilde{D}_v = \frac{\partial}{\partial v} + h_v \frac{\partial}{\partial h} + h_{vt} \frac{\partial}{\partial h_t} + h_{vx} \frac{\partial}{\partial h_x} + h_{vv} \frac{\partial}{\partial h_v} + h_{vv} \frac{\partial}{\partial h_{vt}},
\]

\[
\tilde{D}_4 \equiv \tilde{D}_{v_t} = \frac{\partial}{\partial v_t} + h_{vt} \frac{\partial}{\partial h} + h_{vt} \frac{\partial}{\partial h_t} + h_{vx} \frac{\partial}{\partial h_x} + h_{vv} \frac{\partial}{\partial h_v} + h_{vv} \frac{\partial}{\partial h_{vt}},
\]

Then we use the result of the usual prolongation \( (3.8) \), specifically the expression for \( \zeta_2 \), and calculate the coordinates \( \omega_i \) by the following new prolongation formulae:

\[
\omega_i = \tilde{D}_i(\mu) - h_t \tilde{D}_i(\xi^1) - h_x \tilde{D}_i(\xi^2) - h_v \tilde{D}_i(\eta) - h_{v_t} \tilde{D}_i(\zeta_1) - h_{v_x} \tilde{D}_i(\zeta_2). \quad (3.10)
\]

The infinitesimal invariance test for the system \( (3.6) \) has the form

\[
\left( \zeta_1 - h \zeta_{22} - \mu v_{xx} \right)_{(3.6)} = 0, \quad (3.11)
\]

\[
\omega_i_{(3.6)} = 0, \quad i = 1, \ldots, 4. \quad (3.12)
\]

Taking into account the equations \( (3.6) \) and invoking that \( \mu \) does not depend on the derivatives \( h_x, \ldots, h_{v_t} \), we reduce the differentiations \( (3.9) \) to the following partial derivatives:

\[
\tilde{D}_1 = \frac{\partial}{\partial t}, \quad \tilde{D}_2 = \frac{\partial}{\partial x}, \quad \tilde{D}_3 = \frac{\partial}{\partial v}, \quad \tilde{D}_3 = \frac{\partial}{\partial v_t}.
\]

This simplifies the prolongation formulae \( (3.10) \). Since the functions \( \xi, \eta, \zeta \) do not depend on \( h \), the equations \( (3.12) \) become

\[
\frac{\partial \mu}{\partial t} - h_{v_x} \frac{\partial \zeta_2}{\partial t} = 0, \quad \frac{\partial \mu}{\partial x} - h_{v_x} \frac{\partial \zeta_2}{\partial x} = 0,
\]
\[ \frac{\partial \mu}{\partial v} - h v_x \frac{\partial \zeta_2}{\partial v} = 0, \quad \frac{\partial \mu}{\partial v_t} - h v_x \frac{\partial \zeta_2}{\partial v_t} = 0. \]

Since \( h \) and \( h v_x \) are algebraically independent, the above equations split into the following two systems:

\[ \frac{\partial \mu}{\partial t} = 0, \quad \frac{\partial \mu}{\partial x} = 0, \quad \frac{\partial \mu}{\partial v} = 0, \quad \frac{\partial \mu}{\partial v_t} = 0 \tag{3.13} \]

and

\[ \mu = \mu(v_x, h), \quad \frac{\partial \zeta_2}{\partial t} = 0, \quad \frac{\partial \zeta_2}{\partial x} = 0, \quad \frac{\partial \zeta_2}{\partial v} = 0, \quad \frac{\partial \zeta_2}{\partial v_t} = 0. \tag{3.14} \]

Equations (3.13) yield

\[ \mu = \mu(v_x, h). \tag{3.15} \]

Substituting the expression \( \zeta_2 = \eta_x + v_x \eta_v - v_t v_x \xi_1 - v_t v_x \xi_1^1 - v_x v_x \xi_2 - v_x^2 \xi_2^2 \) (see (3.8)) in (3.14), equating to zero separately the coefficients for different powers in \( v_x \) and \( v_t \), and integrating the resulting equations, we obtain:

\[ \xi_1 = \xi_1^1(t), \quad \xi_2 = A_1(t)x + A_2(t), \quad \eta = A_3(t)v + C_1x + A_4(t), \tag{3.16} \]

where \( A_i(t) \) are arbitrary functions and \( C_i = \text{const} \). Substitution of the expressions (3.15) and (3.16) into (3.11) yields the following general solution to the determining equations (3.11)-(3.12):

\[ \xi_1 = C_1x + C_2, \quad \xi_2 = C_3 + C_4v + C_5, \tag{3.17} \]

\[ \eta = C_6x + C_7v + C_8, \quad \mu = (2C_4v_x + 2C_3 - C - 1)h. \]

Substituting (3.17) in (3.4), one arrives at the following theorem.

**Theorem 3.1.** The equivalence algebra \( L_E \) for the filtration equations (3.1) is an 8-dimensional Lie algebra spanned by

\[ Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = \frac{\partial}{\partial v}, \quad Y_4 = t \frac{\partial}{\partial t} - h \frac{\partial}{\partial h}, \tag{3.18} \]

\[ Y_5 = x \frac{\partial}{\partial x} + 2h \frac{\partial}{\partial h}, \quad Y_6 = v \frac{\partial}{\partial x} + 2v_x h \frac{\partial}{\partial h}, \quad Y_7 = x \frac{\partial}{\partial v}, \quad Y_8 = v \frac{\partial}{\partial v}. \]

The operators (3.18) generate an eight-parameter group. The transformations of this group are obtained by solving the Lie equations for each of the basic generators (3.18) and taking the composition of the resulting one-parameter groups. Note that
since the operator $Y_6$ involves the derivative $v_x$, we should use its first prolongation, namely, extend its action to $v_x$ and write as follows:

$$Y_6 = v \frac{\partial}{\partial x} - v_x^2 \frac{\partial}{\partial v_x} + 2v_x h \frac{\partial}{\partial h}$$

Then, denoting by $a_6$ the parameter of the one-parameter group with the generator $Y_6$, we have the following Lie equations:

$$\frac{d\bar{t}}{da_6} = 0, \quad \frac{d\bar{x}}{da_6} = \bar{v}, \quad \frac{d\bar{v}}{da_6} = 0, \quad \frac{d\bar{v}_x}{da_6} = -\bar{v}_x^2, \quad \frac{d\bar{h}}{da_6} = 2\bar{v}_x \bar{h}.$$ 

The integration, using the initial conditions

$$\bar{t}|_{a_6=0} = t, \quad \bar{x}|_{a_6=0} = x, \quad \bar{v}|_{a_6=0} = v, \quad \bar{v}_x|_{a_6=0} = v_x, \quad \bar{h}|_{a_6=0} = h$$

yields:

$$\bar{t} = t, \quad \bar{x} = x + va_6, \quad \bar{v} = v, \quad \bar{v}_x = \frac{v_x}{1 + a_6 v_x}, \quad \bar{h} = (1 + a_6 v_x)^2 h.$$ 

Whence, ignoring the transformation formula for $v_x$, one obtains the equivalence transformation of the form (3.5). For all other operators (3.18), the integration of the Lie equations is straightforward. We have:

$$Y_1 : \quad \bar{t} = t + a_1, \quad \bar{x} = x, \quad \bar{v} = v, \quad \bar{h} = h;$$

$$Y_2 : \quad \bar{t} = t, \quad \bar{x} = x + a_2, \quad \bar{v} = v, \quad \bar{h} = h;$$

$$Y_3 : \quad \bar{t} = t, \quad \bar{x} = x, \quad \bar{v} = v + a_3, \quad \bar{h} = h;$$

$$Y_4 : \quad \bar{t} = a_4 t, \quad \bar{x} = x, \quad \bar{v} = v, \quad \bar{h} = \frac{1}{a_4} h, \quad a_4 > 0;$$

$$Y_5 : \quad \bar{t} = t, \quad \bar{x} = a_5 x, \quad \bar{v} = v, \quad \bar{h} = a_5^2 h, \quad a_5 > 0;$$

$$Y_6 : \quad \bar{t} = t, \quad \bar{x} = x + a_6 v, \quad \bar{v} = v, \quad \bar{h} = (1 + a_6 v_x)^2 h;$$

$$Y_7 : \quad \bar{t} = t, \quad \bar{x} = x, \quad \bar{v} = v + a_7 x, \quad \bar{h} = h;$$

$$Y_8 : \quad \bar{t} = t, \quad \bar{x} = x, \quad \bar{v} = a_8 v, \quad \bar{h} = h, \quad a_8 > 0.$$ 

Taking the composition of these transformations and setting

$$\alpha = a_4, \quad \beta_1 = a_5, \quad \beta_2 = a_6, \quad \beta_3 = a_5 a_7 a_8, \quad \beta_4 = (1 + a_6 a_7) a_8,$$

$$\gamma_1 = a_1 a_4, \quad \gamma_2 = a_2 a_5 + a_3 a_6, \quad \gamma_3 = (a_3 + a_2 a_5 a_7 + a_3 a_6 a_7) a_8,$$

we arrive at the following result.
Theorem 3.2. The continuous group $E_c$ of equivalence transformations (3.5) for the family of filtration equations (3.1) has the following form:

$$\bar{t} = \alpha t + \gamma_1, \quad \bar{x} = \beta_1 x + \beta_2 v + \gamma_2,$$
$$\bar{v} = \beta_3 x + \beta_4 v + \gamma_3, \quad \bar{h} = (\beta_1 + \beta_2 v_x)^2 \frac{h}{\alpha},$$

(3.19)

where $\alpha, \beta$ and $\gamma$ are constant coefficients obeying the conditions

$$\alpha > 0, \quad \beta_1 > 0, \quad \beta_4 > 0, \quad \beta_1 \beta_4 - \beta_2 \beta_3 > 0.$$  

(3.20)

Note that the last inequality in (3.20) follows from the equation $\beta_1 \beta_4 - \beta_2 \beta_3 = a_5 a_8$.

3.2 Direct search for the equivalence group $E$

Let us outline the direct method. We look for the general equivalence transformations in the form (3.2):

$$\bar{x} = \varphi(t, x, v), \quad \bar{t} = \psi(t, x, v), \quad \bar{v} = \Phi(t, x, v).$$

Under this change of variables, the differentiation operators $D_t, D_x$ are transformed according to the formulas

$$D_t = D_t(\varphi) D_x + D_t(\psi) D_t, \quad D_x = D_x(\varphi) D_x + D_x(\psi) D_t,$$

the use of which leads to the following expression for $\bar{v}_x$:

$$\bar{v}_x = \frac{D_x(\Phi) D_t(\psi) - D_t(\Phi) D_x(\psi)}{D_x(\varphi) D_t(\psi) - D_t(\varphi) D_x(\psi)}.$$

It follows from the definition of equivalence transformations that the right-hand side of the latter equation should depend only on $v_x$ so its derivatives with respect to $t, x, v, v_t$ are equal to zero. This condition leads to a system of equations on the functions $\varphi, \psi, \Phi$ whose solution has the form

$$\varphi = A_1(t)(\beta_1 x + \beta_2 v) + A_2(t),$$
$$\psi = \psi(t),$$
$$\Phi = A_1(t)(\beta_3 x + \beta_4 v) + A_3(t),$$

(3.21)

where $\beta_1 \beta_4 - \beta_2 \beta_3 \neq 0, \psi'(t) \neq 0, A_1(t) \neq 0$.

One can further specify the functions $A_i(t)$ and $\psi(t)$ by substituting (3.21) in Eqs. (3.2)-(3.3) and prove the following statement.
Theorem 3.3. The general equivalence group $\mathcal{E}$ of filtration equations (3.1) has the form (3.19):

$$\tilde{t} = \alpha t + \gamma_1, \quad \tilde{x} = \beta_1 x + \beta_2 v + \gamma_2,$$

$$\tilde{v} = \beta_3 x + \beta_4 v + \gamma_3, \quad \tilde{h} = (\beta_1 + \beta_2 v_x)^2 \frac{h}{\alpha}$$

with arbitrary coefficients $\alpha, \beta_i, \gamma_i$, obeying only the non-degeneracy condition (cf. the conditions (3.20)):

$$\beta_1 \beta_4 - \beta_2 \beta_3 \neq 0. \quad (3.22)$$

Remark 3.1. Theorems 3.2 and 3.3 show that the general equivalence group $\mathcal{E}$ can be obtained from the continuous equivalence group $\mathcal{E}_c$ merely by completing the latter by the reflections $t \to -t$ and $x \to -x$.

3.3 Two equations related to the filtration equation

If we differentiate both sides of the filtration equation (3.1) with respect to $x$ and set

$$u = v_x \quad (3.23)$$

we get the non-linear heat equation

$$u_t = [h(u)u_x]_x. \quad (3.24)$$

The equivalence algebra $L_\mathcal{E}$ for Eq. (3.24) is a 6-dimensional Lie algebra spanned by following generators (cf. (3.18)):

$$Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = t \frac{\partial}{\partial t} - h \frac{\partial}{\partial h},$$

$$Y_4 = x \frac{\partial}{\partial x} + 2h \frac{\partial}{\partial h}, \quad Y_5 = \frac{\partial}{\partial u}, \quad Y_6 = u \frac{\partial}{\partial u}. \quad (3.25)$$

Calculating the transformations of the continuous equivalence group $\mathcal{E}_c$ generated by (3.25) and adding the reflections

$$t \to -t, \quad x \to -x, \quad w \to -w,$$

we arrive at the well-known equivalence group $\mathcal{E}$ for Equation (3.24):

$$\tilde{t} = \alpha t + \gamma_1, \quad \tilde{x} = \beta_1 x + \gamma_2,$$

$$\tilde{u} = \delta_1 u + \delta_2, \quad \tilde{h} = \frac{\beta_1^2}{\alpha} h. \quad (3.26)$$
where \( \alpha \beta_1 \delta_1 \neq 0 \).

Furthermore, by setting
\[
v = w_x
\]
and integrating the filtration equation (3.1) with respect to \( x \) we get the equation
\[
w_t = H(w_2)
\]
The equivalence algebra \( \mathcal{L}_E \) for Eq. (3.28) is a 9-dimensional Lie algebra spanned by
\[
Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = \frac{\partial}{\partial w}, \quad Y_4 = t \frac{\partial}{\partial t} - h \frac{\partial}{\partial h}, \quad Y_5 = x \frac{\partial}{\partial x}, \quad Y_6 = t \frac{\partial}{\partial w} + \frac{\partial}{\partial H}, \quad Y_7 = w \frac{\partial}{\partial w} + H \frac{\partial}{\partial H}, \quad Y_8 = x \frac{\partial}{\partial w}, \quad Y_9 = x^2 \frac{\partial}{\partial w}.
\]
Calculating the transformations of the continuous equivalence group \( \mathcal{E}_c \) generated by (3.29) and adding the reflections \( t \rightarrow -t, x \rightarrow -x, w \rightarrow -w \), we arrive at the following complete equivalence group \( \mathcal{E} \) for Equation (3.28):
\[
\bar{t} = \alpha t + \gamma_1, \quad \bar{x} = \beta_1 x + \gamma_2, \quad \bar{w} = \delta_1 w + \delta_2 x^2 + \delta_3 x + \delta_4 t + \delta_5, \quad H = \frac{\delta_1}{\alpha} H + \delta_4,
\]
where \( \alpha \beta_1 \delta_1 \neq 0 \).

4 Equivalence groups for non-linear wave equations

4.1 The equations \( v_{tt} = f(x, v_x) v_{xx} + g(x, v_x) \)

Consider the family of non-linear wave equations (see [27])
\[
v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)
\]
with two arbitrary functions \( f(x, v_x) \) and \( g(x, v_x) \).

Let us denote \( f = f^1, g = f^2 \) and seek for an operator of the group \( \mathcal{E}_c \) in the form
\[
Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial v} + \mu^k \frac{\partial}{\partial f^k}
\]
from the invariance conditions of Eq. (4.1) written as the system
\[
v_{tt} = f^1 v_{xx} - f^2 = 0, \quad f^k_t = f^k_v = f^k_{v_t} = 0
\]
Here, $v$ and $f^k$ are considered as differential variables: $v$ on the space $(t, x)$ and $f^k$ on the extended space $(t, x, v, v_t, v_x)$. The coordinates $\xi^1, \xi^2, \eta$ of the operator (4.2) are sought as functions of $t, x, v$ while the coordinates, $\mu^k$ are sought as functions of $t, x, v, v_t, v_x, f^1, f^2$. The invariance conditions of the system (4.3) are

\[
\tilde{Y}(v_{tt} - f^1 v_{xx} - f^2) = 0 \quad (4.4)
\]

\[
\tilde{Y}(f^k_t) = \tilde{Y}(f^k_v) = \tilde{Y}(f^k_{v_t}) = 0 \quad (k = 1, 2),
\]

where $\tilde{Y}$ is the prolongation of the operator (4.2):

\[
\tilde{Y} = Y + \zeta_1 \frac{\partial}{\partial v_t} + \zeta_2 \frac{\partial}{\partial v_x} + \zeta_{11} \frac{\partial}{\partial v_{tt}} + \zeta_{22} \frac{\partial}{\partial v_{xx}} + \omega_1^k \frac{\partial}{\partial f^k_t} + \omega_0^k \frac{\partial}{\partial f^k_v} + \omega_{01}^k \frac{\partial}{\partial f^k_{v_t}}.
\]

The coefficients $\zeta$ are given by the usual prolongation formulæ:

\[
\begin{align*}
\zeta_1 &= D_t(\eta) - v_t D_t(\xi^1) - v_x D_t(\xi^2), \\
\zeta_2 &= D_x(\eta) - v_t D_x(\xi^1) - v_x D_x(\xi^2), \\
\zeta_{11} &= D_t(\zeta_1) - v_{tt} D_t(\xi^1) - v_{tx} D_t(\xi^2), \\
\zeta_{22} &= D_x(\zeta_2) - v_{tx} D_x(\xi^1) - v_{xx} D_x(\xi^2),
\end{align*}
\]

whereas the coefficients $\omega$ are obtained by applying the secondary prolongation procedure (see Section 3.1) to the differential variables $f^k$ with the independent variables $(t, x, v, v_t, v_x)$. Namely,

\[
\omega_1^k = \tilde{D}_t(\mu^k) - f^k_t \tilde{D}_t(\xi^1) - f^k_x \tilde{D}_t(\xi^2) - f^k_v \tilde{D}_t(\eta) - f^k_{v_t} \tilde{D}_t(\zeta_1) - f^k_{v_x} \tilde{D}_t(\zeta_2),
\]

where $\tilde{D}_t$ has the form

\[
\tilde{D}_t = \frac{\partial}{\partial t} + f^k_t \frac{\partial}{\partial f^k}
\]

and in view of Eqs. (4.3) reduces to

\[
\tilde{D}_t = \frac{\partial}{\partial t}.
\]

Furthermore, we obtain $\omega_0^k$ and $\omega_{01}^k$ by replacing in Eq. (4.6) the operator $\tilde{D}_t$ successively by the operators

\[
\begin{align*}
\tilde{D}_v &= \frac{\partial}{\partial v} + f^k_v \frac{\partial}{\partial f^k}, \\
\tilde{D}_{v_t} &= \frac{\partial}{\partial v_t} + f^k_{v_t} \frac{\partial}{\partial f^k}
\end{align*}
\]
and noting that in view of Eqs. (4.3) we have
\[ \tilde{D}_v = \frac{\partial}{\partial v}, \quad \tilde{D}_{vt} = \frac{\partial}{\partial v_t}. \]

Finally, we have:
\[ \omega_{11} = \mu_1 - f_1 \xi_1 - f_2 (\zeta_2)_t, \]
\[ \omega_0 = \mu_0 - f_1 \xi_2 - f_2 (\zeta_2)_v, \]
\[ \omega_{01} = \mu_0 - f_1 (\zeta_2)_{v_t}. \]  

Finally, we have:
\[ \omega_{11} = \omega_0 = \omega_{01} = 0, \quad k = 1, 2. \]  

The invariance conditions (4.5) have the form
\[ \omega_{11} = \omega_0 = \omega_{01} = 0, \quad k = 1, 2. \]  

Substituting the expressions (4.7) and noting that Eqs. (4.8) hold for arbitrary values of \( f^1 \) and \( f^2 \), we obtain
\[ \mu_1 = \mu_0 = \mu_{01} = 0, \quad \xi_1 = \xi_2 = 0, \]
\[ (\zeta_2)_t = (\zeta_2)_v = (\zeta_2)_{v_t} = 0. \]

Integration yields:
\[ \xi^1 = \xi^1(t), \quad \xi^2 = \xi^2(x), \]
\[ \eta = c_1 v + F(x) + H(t), \]
\[ \mu^k = \mu^k(x, v_x, f^1, f^2). \]  

Now we write the invariance condition (4.4):
\[ \zeta_{11} - \mu^1 v_{xx} - f^1 \zeta_{22} - \mu^2 = 0, \]

take into account Eqs. (4.9) and obtain:
\[ (\xi^1)'v_t + \{[C_1 - 2(\xi^1)']f^1 - \mu^1 - [C_1 - 2(\xi^2)']f^1\} v_{xx} + [C_1 - 2(\xi^1)']f^2 + H'' - f^1 F'' + f^1 v_x (\xi^2)'' - \mu^2 = 0. \]

Since \( v, v_t, v_x \) and \( v_{xx} \) are independent variables, it follows:
\[ \xi^1 = C_2 t + C_3, \quad \xi^2 = \varphi(x), \]
\[ \eta = C_1 v + F(x) + C_4 t^2 + C_5 t, \]
\[ \mu^1 = 2(\varphi' - C_2) f, \]
\[ \mu^2 = (C_1 - 2C_2) g + 2C_4 + (\varphi'' v_x - F'') f, \]

where \( C_1, C_2, C_3, C_4, C_5 \) are arbitrary constants, and \( \varphi(x) \) and \( F(x) \) arbitrary functions. Thus, we have the following result.
Theorem 4.1. The family of non-linear wave equations (4.1) has an infinite equivalence group $\mathcal{E}_c$. The corresponding Lie algebra $L_{\mathcal{E}}$ is spanned by the generators

\[
Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial v}, \quad Y_3 = t \frac{\partial}{\partial v}, \\
Y_4 = x \frac{\partial}{\partial v}, \quad Y_5 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + 2v \frac{\partial}{\partial v}, \\
Y_6 = t \frac{\partial}{\partial t} - 2f \frac{\partial}{\partial f} - 2g \frac{\partial}{\partial g}, \quad Y_7 = t^2 \frac{\partial}{\partial v} + 2 \frac{\partial}{\partial g},
\]

(4.11)

Remark 4.1. The general equivalence group $\mathcal{E}$ contains, along with the continuous subgroup $\mathcal{E}_c$, also three independent reflections:

\[
t \mapsto -t, \\
x \mapsto -x, \\
v \mapsto -v, \quad g \mapsto -g.
\]

(4.12) (4.13) (4.14)

4.2 The equations $u_{tt} - u_{xx} = f(u, u_t, u_x)$

Similar calculations for the non-linear wave equations of the form

\[
u_{tt} - u_{xx} = f(u, u_t, u_x) \quad (f \neq 0).
\]

(4.15)

provide the infinite-dimensional equivalence algebra $L_{\mathcal{E}}$ spanned by ([28])

\[
Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = x \frac{\partial}{\partial x} + t \frac{\partial}{\partial u_t} - u_x \frac{\partial}{\partial u_x},
\]

\[
Y_4 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2f \frac{\partial}{\partial f} - u_t \frac{\partial}{\partial u_t} - u_x \frac{\partial}{\partial u_x},
\]

\[
Y_\varphi = \varphi(x) \frac{\partial}{\partial u} + [\varphi' f + \varphi''(u_t^2 - u_x^2)] \frac{\partial}{\partial f} + \varphi' u_t \frac{\partial}{\partial u_t} + \varphi' u_x \frac{\partial}{\partial u_x},
\]

where $\varphi = \varphi(x)$ is an arbitrary function.

5 Equivalence groups for evolution equations

5.1 The generalised Burgers equation

The generalised Burgers equation

\[
u_t + u u_x + f(t) u_{xx} = 0,
\]

(5.1)
Nail H. Ibragimov

has applications in acoustic phenomena. It has been also used to model turbulence and certain steady state viscous flows. The group $E$ of equivalence transformations for Eqs. (5.1) was calculated in [29] (see also [30]) by the direct method. The group $E$ comprises the linear transformation:

$$\bar{x} = c_3c_5x + c_1c_5^2t + c_2, \quad \bar{t} = c_5^2t + c_4, \quad \bar{u} = \frac{c_3}{c_5}u + c_1 \quad (5.2)$$

and the projective transformation:

$$\bar{x} = \frac{c_3c_6x - c_1}{c_6^2t - c_4} + c_2, \quad \bar{t} = c_5 - \frac{1}{c_6^2t - c_4}, \quad \bar{u} = c_3c_6(ut - x) + \frac{c_3c_4}{c_6}u + c_1, \quad (5.3)$$

where $c_1, \ldots, c_6$ are constants such that $c_3 \neq 0$, $c_5 \neq 0$, and $c_6 \neq 0$. Under these transformations, the coefficient $f(t)$ of the Eq. (5.1) is mapped to

$$\bar{f} = c_3^2 f. \quad (5.4)$$

It is manifest from Eqs. (5.2) and (5.3) that the continuous equivalence group $E_\epsilon$ is generated by the six-dimensional equivalence algebra $L_\epsilon$ spanned by

$$Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, \quad Y_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u},$$

$$Y_5 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + 2f\frac{\partial}{\partial f}, \quad Y_6 = t^2\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} + (x - ut)\frac{\partial}{\partial u}. \quad (5.5)$$

5.2 The equations $u_t = f(x, u)u_{xx} + g(x, u, u_x)$

In [31], we considered the equivalence group and calculated the invariants for the family of evolution equations of the form

$$u_t = f(x, u)u_{xx} + g(x, u, u_x). \quad (5.6)$$

A number of particular cases of this class of equations have been used to model physical problems. Such examples are the well-known nonlinear diffusion equation

$$u_t = [D(u)u_x]_x,$$

and its modifications, e.g. equations of the form

$$u_t = [g(x)D(u)u_x]_x,$$

$$u_t = [g(x)D(u)u_x]_x - K(u)u_x,$$

$$u_t = (u^n)_{xx} + g(x)u^m + f(x)u^4u_x.$$
The generalised Burgers equation (5.1) is also a particular case of Eq. (5.6).

The class of equations (5.6) has an infinite continuous equivalence group $E_c$ generated by the infinite-dimensional Lie algebra $L_E$ spanned by the operators

$$Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = t \frac{\partial}{\partial t} - f \frac{\partial}{\partial f} - g \frac{\partial}{\partial g},$$

$$Y_\phi = \phi(x) \frac{\partial}{\partial x} - \phi' u_x \frac{\partial}{\partial u_x} + 2 \phi' f \frac{\partial}{\partial f} + \phi'' f u_x \frac{\partial}{\partial g},$$

$$Y_\psi = \psi(x, u) \frac{\partial}{\partial u} + \left( \psi_x + \psi_x u_x \right) \frac{\partial}{\partial u_x} + \left[ \psi_u g - (\psi_{uu} u_x^2 + 2 \psi_{xx} u_x + \psi_{xx}) f \right] \frac{\partial}{\partial g}.$$ 

### 5.3 A model from tumour biology

The system of equations

$$u_t = f(u) - (uc_x)_x,$$

$$c_t = -g(c, u),$$

(5.7)

where $f(u)$ and $g(c, u)$ are, in general, arbitrary functions, are used in mathematical biology for describing spread of malignant tumour.

The equivalence transformations for Eqs. (5.7) are calculated in [32]. It is shown that the system (5.7) has the six-dimensional equivalence algebra spanned by the following generators:

$$Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2f \frac{\partial}{\partial f} - 2g \frac{\partial}{\partial g},$$

$$Y_4 = \frac{\partial}{\partial c}, \quad Y_5 = x \frac{\partial}{\partial x} + 2c \frac{\partial}{\partial c} + 2g \frac{\partial}{\partial g}, \quad Y_6 = u \frac{\partial}{\partial u} + f \frac{\partial}{\partial f}.$$ 

### 6 Examples from non-linear acoustics

The equation

$$\frac{\partial}{\partial t} \left( \frac{\partial u}{\partial x} - u \frac{\partial u}{\partial t} \right) = -\beta u, \quad \beta = \text{const} \neq 0$$

is used in non-linear acoustics for describing several physical phenomena. The following two generalizations of this model and their equivalence groups were given in [33] in accordance with our principle of a priori use of symmetries. The first generalized model has the form

$$\frac{\partial}{\partial t} \left[ \frac{\partial u}{\partial x} - P(u) \frac{\partial u}{\partial t} \right] = F(x, u).$$ 

(6.1)
Its equivalence algebra $L_e$ is a seven-dimensional Lie algebra spanned by

\[ Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = \frac{\partial}{\partial u}, \quad Y_4 = t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u} + P \frac{\partial}{\partial P}, \]
\[ Y_5 = u \frac{\partial}{\partial u} + F \frac{\partial}{\partial F}, \quad Y_6 = x \frac{\partial}{\partial t} - \frac{\partial}{\partial P}, \quad Y_7 = x \frac{\partial}{\partial x} - P \frac{\partial}{\partial P} - F \frac{\partial}{\partial F}. \] (6.2)

The second generalized model has the form

\[ \frac{\partial}{\partial t} \left[ \frac{\partial u}{\partial x} - Q(x, u) \frac{\partial u}{\partial t} \right] = F(x, u). \] (6.3)

Its equivalence algebra is infinite-dimensional and comprises the operators

\[ Y_1 = \varphi(x) \frac{\partial}{\partial t} - \varphi'(x) \frac{\partial}{\partial Q}, \quad Y_2 = \psi(x) \frac{\partial}{\partial x} - \psi'(x) Q \frac{\partial}{\partial Q} - \psi'(x) F \frac{\partial}{\partial F}, \]
\[ Y_3 = \lambda(x) \frac{\partial}{\partial u}, \quad Y_4 = t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u} + Q \frac{\partial}{\partial Q}, \quad Y_5 = u \frac{\partial}{\partial u} + F \frac{\partial}{\partial F}. \] (6.4)
7 Invariants of linear ordinary differential equations

Here, the method of calculation of invariants will be illustrated by the third-order equation (see Section 2.3)

\[ y''' + 3c_1(x)y'' + 3c_2(x)y' + c_3(x)y = 0. \]  

(7.1)

**Definition 7.1.** A function

\[ h = h(x, y, c, c', c'', \ldots) \]  

(7.2)

of the variables \( x, y \) and of the coefficients \( c = (c_1, c_2, c_3) \) together with their derivatives \( c', c'', \ldots \) of a finite order is called an invariant of the family of linear equations (7.1) if \( h \) is a differential invariant for the equivalence transformations (2.18)-(2.19).

We call \( h \) a semi-invariant if it is invariant only under the subgroup comprising the linear transformation (2.18).

The order of the invariant is the highest order of derivatives \( c', c'', \ldots \) involved in \( h \).

**Remark 7.1.** The independent variable \( x \) is manifestly semi-invariant. Therefore, we can ignore it in calculating semi-invariants.

**Theorem 7.1.** Equation (7.1) has two independent semi-invariants of the first order:

\[ h_1 = c_2 - c_1^2 - c_1', \]  

(7.3)

\[ h_2 = c_3 - 3c_1c_2 + 2c_1^3 + 2c_1c_1' - c_2'. \]  

(7.4)

Any semi-invariant is a function of \( x, y \) and \( h_1, h_2, h_1', h_2', \ldots \).

**Proof.** First we check that there are no semi-invariants of the order 0, i.e. of the form

\[ h = h(y, c_1, c_2, c_3). \]

We use the equivalence generator (2.29) and write the invariance test \( Y_\eta(h) = 0 : \)

\[ \eta \frac{\partial h}{\partial y} + \eta' \frac{\partial h}{\partial c_1} + (\eta'' + 2c_1 \eta') \frac{\partial h}{\partial c_2} + (\eta''' + 3c_1 \eta'' + 3c_2 \eta') \frac{\partial h}{\partial c_3} = 0. \]  

(7.5)

Since the function \( \eta(x) \) is arbitrary, and hence there are no relations between its derivatives, Eq. (7.5) splits into four equations obtained by annulling separately the terms with \( \eta, \eta'', \eta''' \) and \( \eta' \):

\[ \frac{\partial h}{\partial y} = 0, \quad \frac{\partial h}{\partial c_3} = 0, \quad \frac{\partial h}{\partial c_2} + 3c_1 \frac{\partial h}{\partial c_3} = 0, \quad \frac{\partial h}{\partial c_1} + 2c_1 \frac{\partial h}{\partial c_2} + 3c_2 \frac{\partial h}{\partial c_3} = 0. \]

\(^1\)It means that \( h \) is invariant under the equivalence group prolonged to the derivatives \( c', c'', \ldots \).

\(^2\)One can consider other semi-invariants by taking, instead of (2.18), any subgroups of the general equivalence group. Note, that the classical papers [10], [12], [6], [13], [14], [15], [16] mentioned in Introduction deal exclusively with invariants of the subgroup (2.18).
It follows that $h = \text{const.}$, i.e. there are no differential invariants of the order 0.

Now we take the first prolongation of $Y_\eta$ and solve the equation

$$
\eta \frac{\partial h}{\partial y} + \eta' \frac{\partial h}{\partial c_1} + (\eta'' + 2c_1\eta') \frac{\partial h}{\partial c_2} + (\eta''' + 3c_1\eta'' + 3c_2\eta') \frac{\partial h}{\partial c_3} + \eta'' \frac{\partial h}{\partial c_4}
$$

$$
+ (\eta'' + 2c_1\eta'' + 2c_1\eta') \frac{\partial h}{\partial c_5} + (\eta^{(iv)} + 3c_1\eta''' + 3c_2\eta'' + 3c_2\eta') \frac{\partial h}{\partial c_6} = 0
$$

by letting

$$
h = h(y, c_1, c_2, c_3, c_4, c_5, c_6).
$$

Again, the term with $\eta$ yields that $h$ does not depend upon $y$, whereas the term with $\eta^{(iv)}$ yields $\partial h/\partial c_5 = 0$. The terms with $\eta'', \eta', \eta'$ give three linear partial differential equations for the function $h = h(c_1, c_2, c_3, c_4, c_5, c_6)$. These equations have precisely two functionally independent solutions given in (7.3) - (7.4).

We have to continue by taking the second prolongation of $Y_\eta$ and considering the second-order semi-invariants, i.e. letting $h = h(c_1, c_2, c_3, c_4, c_5, c_6)$. However, one can verify that the equation $Y_\eta(h) = 0$, where $Y_\eta$ is the twice-prolonged operator, has precisely four functionally independent solutions. Since $h_1$ and $h_2$ together with their first derivatives provide four functionally independent solutions of this type, the theorem is proved for semi-invariants of the second order. The iteration completes the proof.

**Remark 7.2.** The semi-invariants for third-order equations were calculated by Laguerre (see [6]). He found the first-order semi-invariant (7.3) and a second-order one, $\tilde{h}_2 = c_3 - 3c_1c_2 + 2c_1^2 - c_1''$, instead of (7.3). They are equivalent, namely, $\tilde{h}_2 = h_2 + h_1$.

Let us turn now to the proper invariants. The infinitesimal transformation (2.19),

$$
\bar{x} \approx x + \varepsilon \xi(x),
$$

implies the following infinitesimal transformations of derivatives:

$$
y' \approx (1 + \varepsilon \xi') \bar{y}',
$$

$$
y'' \approx (1 + 2\varepsilon \xi') \bar{y}'' + \varepsilon \bar{y}' \xi'',
$$

$$
y''' \approx (1 + 3\varepsilon \xi') \bar{y}''' + 3\varepsilon \bar{y}'' \xi'' + \varepsilon \bar{y}' \xi'''.
$$

Consequently Eq. (7.1) becomes

$$
\bar{y}''' + 3\bar{c}_1 \bar{y}'' + 3\bar{c}_2 \bar{y}' + \bar{c}_3 \bar{y} = 0,
$$

where

$$
\bar{c}_1 \approx c_1 + \varepsilon (\xi'' - c_1 \xi'),
$$

$$
\bar{c}_2 \approx c_2 + \varepsilon \left(\frac{1}{3} \xi''' + c_1 \xi'' - 2c_2 \xi'\right),
$$

$$
\bar{c}_3 \approx c_3 - 3\varepsilon c_3 \xi'.
$$
The corresponding group generator, extended to the first derivatives of \( c_1, c_2 \) is

\[
Y = \xi \frac{\partial}{\partial x} + (\xi'' - c_1 \xi') \frac{\partial}{\partial c_1} + \left( \frac{1}{3} \xi''' + c_1 \xi'' - 2c_2 \xi' \right) \frac{\partial}{\partial c_2} - 3c_3 \xi' \frac{\partial}{\partial c_3}
\]

\[
+ (\xi'' - c_1 \xi' - 2c_1 \xi') \frac{\partial}{\partial c_1} + \left( \frac{1}{3} \xi^{(iv)} + c_1 \xi'' + c_1' \xi'' - 2c_2 \xi'' - 3c_2 \xi' \right) \frac{\partial}{\partial c_2}.
\]

We rewrite it in terms of the semi-invariants (7.3)- (7.4) and after prolongation obtain:

\[
Y = \xi \frac{\partial}{\partial x} - \left[ \frac{2}{3} \xi'' + 2h_1 \xi' \right] \frac{\partial}{\partial h_1} - \left[ \frac{1}{3} \xi^{(iv)} + h_1 \xi'' + 3h_2 \xi' \right] \frac{\partial}{\partial h_2}
\]

\[
- \left[ \frac{2}{3} \xi^{(iv)} + 2h_1 \xi'' + 3h_1' \xi' \right] \frac{\partial}{\partial h_1'} - \left[ \frac{2}{3} \xi^{(v)} + 2h_1 \xi''' + 5h_1' \xi'' + 4h_1'' \xi' \right] \frac{\partial}{\partial h_1''}
\]

\[
- \left[ \frac{1}{3} \xi^{(v)} + h_1 \xi''' + h_1' \xi'' + 3h_2 \xi'' + 4h_2' \xi' \right] \frac{\partial}{\partial h_2'} + \ldots .
\]

(7.6)

The following results were obtained in [4] (see also [22], Section 10.2) by applying to the operator (7.6) the approach used in the proof of Theorem 7.1.

**Theorem 7.2.** Eq. (7.1) has a singular invariant equation with respect to the group of general equivalence transformations, namely, the equation

\[
h_1' - 2h_2 = 0,
\]

(7.7)

where \( h_1 \) and \( h_2 \) are the semi-invariants (7.3) and (7.4), respectively.

**Theorem 7.3.** The least invariant of equation (7.1), i.e. an invariant involving the derivatives of \( h_1 \) and \( h_2 \) of the lowest order is

\[
\theta = \frac{1}{\lambda^2} \left[ 7 \left( \frac{\lambda'}{\lambda} \right)^2 - 6 \frac{\lambda''}{\lambda} + 27h_1 \right]^3,
\]

(7.8)

where

\[
\lambda = h_1' - 2h_2.
\]

(7.9)

The higher-order invariants are obtained from \( \theta \) by means of invariant differentiation, and any invariant of an arbitrary order is a function of \( \theta \) and its invariant derivatives.

**Corollary 7.1.** Eq. (7.1) is equivalent to the equation

\[
y'' = 0
\]

if and only if \( \lambda = 0 \), i.e. the invariant equation (7.7) holds (see [6]).
Corollary 7.2. The necessary and sufficient condition for Eq. (7.1) to be equivalent to

\[ y''' + y = 0 \]

is that \( \lambda \neq 0 \) and that \( \theta = 0 \). For example, the equation

\[ y''' + c(x)y = 0 \]

is equivalent to

\[ z''' + z = 0 \]

only in the case

\[ c(x) = (kx + l)^{-6}, \]

where the constants \( k \) and \( l \) do not vanish simultaneously.

8 Invariants of hyperbolic second-order linear partial differential equations in two variables

In this section and sections 9, 10 we will discuss the invariants for all three types of equations, hyperbolic, elliptic and parabolic, with two independent variables. The calculations are based on my recent works [34], [8] and illustrate the application of our method to partial differential equations with infinite equivalence groups.

8.1 Equivalence transformations

Consider the general hyperbolic equation written in the characteristic variables \( x, y \), i.e. in the following standard form:

\[ u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0. \] (8.1)

Recall that an equivalence transformation of equations (8.1) is defined as an invertible transformation

\[ \bar{x} = f(x, y, u), \quad \bar{y} = g(x, y, u), \quad \bar{u} = h(x, y, u) \] (8.2)

such that the equation (8.1) with any coefficients \( a, b, c \) remains linear and homogeneous but the transformed equation can have, in general, new coefficients \( \bar{a}, \bar{b}, \bar{c} \). Two equations of the form (8.1) are are said to be equivalent if they can be connected by a properly chosen equivalence transformation. Proceeding as in Sections 3.1 and 4.1, we prove the following result.
Theorem 8.1. The equivalence algebra $L_E$ for Eqs. (8.1) comprises the operators

$$Y = \xi(x) \frac{\partial}{\partial x} + \eta(y) \frac{\partial}{\partial y} + \sigma(x, y) u \frac{\partial}{\partial u} + \mu^1 \frac{\partial}{\partial a} + \mu^2 \frac{\partial}{\partial b} + \mu^3 \frac{\partial}{\partial c}, \tag{8.3}$$

where $\xi(x), \eta(y), \sigma(x, y)$ are arbitrary functions, and $\mu^i$ are given by

$$\begin{align*}
\mu^1 &= -(\sigma_y + a \eta'), \\
\mu^2 &= -(\sigma_x + b \xi'), \\
\mu^3 &= -(\sigma_{xy} + a \sigma_x + b \sigma_y + c \eta' + c \xi').
\end{align*} \tag{8.4}$$

The operator (8.3) generates the continuous infinite group $E_c$ of equivalence transformations (8.2) composed of the linear transformation of the dependent variable:

$$\bar{u} = \phi(x, y) u, \quad \phi(x, y) \neq 0, \tag{8.5}$$

and invertible changes of the independent variables of the form:

$$\bar{x} = f(x), \quad \bar{y} = g(y), \tag{8.6}$$

where $\phi(x, y), f(x)$ and $g(y)$ are arbitrary functions such that $f'(x) \neq 0, g'(y) \neq 0$.

Remark 8.1. The general group $E$ of equivalence transformations (8.2) for the family of hyperbolic equations (8.1) consists of the continuous group $E_c$ augmented by the interchange of the variables,

$$x_1 = y, \quad y_1 = x. \tag{8.7}$$

Hence, the group $E$ contains, along with (8.6), the change of variables

$$\tilde{x} = r(y), \quad \tilde{y} = s(x) \tag{8.8}$$

obtained by taking the composition of (8.6) and (8.7).

8.2 Semi-invariants

In what follows, we will use only the continuous equivalence group of transformations (8.5)-(8.5) written in the form:

$$u = \varphi(x, y) v, \quad \varphi(x, y) \neq 0, \tag{8.9}$$

$$\bar{x} = f(x), \quad \bar{y} = g(y), \tag{8.10}$$

where $v = v(\bar{x}, \bar{y})$ is a new dependent variable.
Definition 8.1. A function
\[ J = J(x, y, a, b, c, a_x, a_y, \ldots) \] (8.11)
is called an invariant of the family of hyperbolic equations (8.1) if it is a differential invariant for the equivalence group (8.5)-(8.5). We call \( J \) a semi-invariant if it is invariant only under the linear transformation (8.9) of the dependent variable.

Let us find all semi-invariants. The apparent semi-invariants \( x \) and \( y \) are not considered in further calculations. One can proceed by using directly the operator (8.3) by letting \( \xi = \eta = 0 \), but one does not need to remember the expressions (8.4) for the coefficients \( \mu \). Indeed, we consider the infinitesimal transformation (8.9) by letting
\[ \varphi(x, y) \approx 1 + \varepsilon \sigma(x, y), \]
where \( \varepsilon \) is a small parameter. Thus, we have:\[ u \approx [1 + \varepsilon \sigma(x, y)]v. \] (8.12)
The transformation of derivatives is written, in the first order of precision in \( \varepsilon \), as follows:
\[ u_x \approx (1 + \varepsilon \sigma)v_x + \varepsilon \sigma_x v, \quad u_y \approx (1 + \varepsilon \sigma)v_y + \varepsilon \sigma_y v, \]
\[ u_{xy} \approx (1 + \varepsilon \sigma)v_{xy} + \varepsilon \sigma_y v_x + \varepsilon \sigma_x v_y + \varepsilon \sigma_{xy} v. \] (8.13)
Therefore,
\[ u_{xy} + au_x + bu_y + cu \approx (1 + \varepsilon \sigma)v_{xy} + \varepsilon \sigma_y v_x + \varepsilon \sigma_x v_y + \varepsilon \sigma_{xy} v \]
\[ + (1 + \varepsilon \sigma)av_x + \varepsilon \sigma_x av + (1 + \varepsilon \sigma)bv_y + \varepsilon \sigma_y bv(1 + \varepsilon \sigma)cv, \]
whence the infinitesimal transformation of the equation (8.1):
\[ v_{xy} + (a + \varepsilon \sigma_y) v_x + (b + \varepsilon \sigma_x) v_y + [c + \varepsilon(\sigma_{xy} + a\sigma_x + b\sigma_y)] v = 0. \]
Thus, the coefficients of the equation (8.1) undergo the infinitesimal transformations
\[ \bar{a} \approx a + \varepsilon \sigma_y, \quad \bar{b} \approx b + \varepsilon \sigma_x, \quad \bar{c} \approx c + \varepsilon(\sigma_{xy} + a\sigma_x + b\sigma_y), \] (8.14)
that provide the generator (cf. 8.3))
\[ Z = \sigma_y \frac{\partial}{\partial a} + \sigma_x \frac{\partial}{\partial b} + (\sigma_{xy} + a\sigma_x + b\sigma_y) \frac{\partial}{\partial c}. \] (8.15)
Let us first consider the functions (8.11) of the form \( J = J(a, b, c) \). Then the infinitesimal invariant test \( Z(J) = 0 \) is written:
\[ \sigma_y \frac{\partial J}{\partial a} + \sigma_x \frac{\partial J}{\partial b} + (\sigma_{xy} + a\sigma_x + b\sigma_y) \frac{\partial J}{\partial c} = 0. \]
Since the function $\sigma(x, y)$ is arbitrary, the latter equation splits into the following three equations obtained by annulling separately the terms with $\sigma_{xy}, \sigma_x$ and $\sigma_y$:
\[
\frac{\partial J}{\partial c} = 0, \quad \frac{\partial J}{\partial b} = 0, \quad \frac{\partial J}{\partial a} = 0.
\]
Thus, there are no invariants $J(a, b, c)$ other than $J = \text{const}$.

Therefore, one should consider, as the next step, the semi-invariants involving first-order derivatives of the coefficients $a, b, c$, i.e. the (8.11) of the form
\[
J = J(a, b, c, a_x, a_y, b_x, b_y, c_x, c_y).
\]
Accordingly, we take the first prolongation of the generator (8.15):
\[
Z = \sigma_y \frac{\partial}{\partial a} + \sigma_x \frac{\partial}{\partial b} + \left( \sigma_{xy} + a \sigma_x + b \sigma_y \right) \frac{\partial}{\partial c} + \sigma_{xy} \frac{\partial}{\partial a_x} + \sigma_{yy} \frac{\partial}{\partial a_y} + \sigma_{xx} \frac{\partial}{\partial b_x} + \sigma_{xy} \frac{\partial}{\partial b_y} + \left( \sigma_{xxy} + a \sigma_{xx} + a_x \sigma_x + b \sigma_{xy} + b_x \sigma_y \right) \frac{\partial}{\partial c_x} + \left( \sigma_{xyy} + a \sigma_{xy} + a_y \sigma_x + b \sigma_{yy} + b_y \sigma_y \right) \frac{\partial}{\partial c_y}.
\]
The equation $Z(J) = 0$, upon equating to zero at first the terms with $\sigma_{xxy}, \sigma_{xyy}$ and then with $\sigma_{xx}, \sigma_{yy}$, yields $\partial J/\partial c_x = 0, \partial J/\partial c_y = 0$ and $\partial J/\partial b_x = 0, \partial J/\partial a_y = 0$, respectively. Hence, $J = J(a, b, c, a_x, b_y)$. Now the terms with $\sigma_{xy}, \sigma_x$ and $\sigma_y$ provide the following system of three equations:
\[
\frac{\partial J}{\partial c} + \frac{\partial J}{\partial a_x} + \frac{\partial J}{\partial b_y} = 0, \quad \frac{\partial J}{\partial b} + a \frac{\partial J}{\partial c} = 0, \quad \frac{\partial J}{\partial a} + b \frac{\partial J}{\partial c} = 0.
\]
One can readily solve the last two equations of this system to obtain $J = J(\lambda, a_x, b_y)$, where $\lambda = ab - c$. Then the first equation of the system yields:
\[
\frac{\partial J}{\partial a_x} + \frac{\partial J}{\partial b_y} - \frac{\partial J}{\partial \lambda} = 0.
\]
The latter equation has two functionally independent solutions, e.g.
\[
J_1 = a_x - b_y, \quad J_2 = a_x + \lambda \equiv a_x + ab - c.
\]
Denoting $h = J_2$ and $k = J_2 - J_1$, one obtains two independent semi-invariants of the equation (8.1), namely, the Laplace invariants:
\[
h = a_x + ab - c, \quad k = b_y + ab - c.
\]

One can verify, by considering the higher-order prolongations, that the semi-invariants involving higher-order derivatives of $a, b, c$ are obtained merely by differentiating the Laplace invariants (8.17) thus proving the following.

**Theorem 8.2.** The general semi-invariant for Eqs. (8.1) has the following form:
\[
J = J(x, y, h, k, h_x, h_y, k_x, k_y, h_{xx}, h_{xy}, h_{yy}, k_{xx}, k_{xy}, k_{yy}, \ldots).
\]
8.3 Laplace’s problem. Calculation of invariants

Laplace’s problem: Find all invariants for the family of the hyperbolic equations (8.1).

Here we will discuss the solution of Laplace’s problem given in [8]. An arbitrary invariant of equation (8.1) is obtained by subjecting the general semi-invariant (8.18) to the invariance test under the changes (8.10) of the independent variables.

The infinitesimal transformation (8.10) of the variable \(x\) has the form

\[
x \approx x + \varepsilon \xi(x)
\]

and yields:

\[
\begin{align*}
    u_x & \approx (1 + \varepsilon \xi') u_x, \\
    u_y & = u_y, \\
    u_{xy} & \approx (1 + \varepsilon \xi') u_{xy},
\end{align*}
\]

where \(\xi' = d\xi(x)/dx\). Hence, equation (8.1) undergoes the infinitesimal transformation

\[
(1 + \varepsilon \xi')u_{xy} + a(1 + \varepsilon \xi')u_x + bu_y + cu = 0,
\]

and can be written, in the first order of precision in \(\varepsilon\), in the form (8.1):

\[
u_{xy} + au_x + (b - \varepsilon \xi' b) u_y + (c - \varepsilon \xi' c) u = 0.
\]

It provides the infinitesimal transformation of the coefficients of equation (8.1):

\[
\begin{align*}
    a & \approx a, \\
    b & \approx b - \varepsilon \xi' b, \\
    c & \approx c - \varepsilon \xi' c.
\end{align*}
\]

The infinitesimal transformations (8.19) and (8.20) define the generator (cf. 8.3))

\[
X = -\xi(x) \frac{\partial}{\partial x} + \xi' b \frac{\partial}{\partial b} + \xi' c \frac{\partial}{\partial c}.
\]

The prolongation of the generator (8.21) to \(a_x\) and \(b_y\) has the form

\[
X = -\xi(x) \frac{\partial}{\partial x} + \xi'(x) \left[ b \frac{\partial}{\partial b} + c \frac{\partial}{\partial c} + a_x \frac{\partial}{\partial a_x} + b_y \frac{\partial}{\partial b_y} \right]
\]

and furnishes the following action on Laplace’s invariants:

\[
X = -\xi(x) \frac{\partial}{\partial x} + \xi'(x) \left[ h \frac{\partial}{\partial h} + k \frac{\partial}{\partial k} \right].
\]

Now we will look for the invariants (8.18) involving the derivatives of \(h\) and \(k\) up to second order. Therefore, we apply the usual prolongation procedure and obtain the following second-order prolongation of the operator (8.22):

\[
X = -\xi(x) \frac{\partial}{\partial x} + \xi' h \frac{\partial}{\partial h} + \xi' k \frac{\partial}{\partial k} + (\xi'' h_x + 2\xi' h_x) \frac{\partial}{\partial h_x} + (\xi'' k_x + 2\xi' k_x) \frac{\partial}{\partial k_x}
\]
Equivalence groups and invariants of linear and non-linear equations

\[ + \xi' h_y \frac{\partial}{\partial h_y} + \xi' k_y \frac{\partial}{\partial k_y} + (\xi''' h + 3 \xi'' h_x + 3 \xi' h_{xx}) \frac{\partial}{\partial h_{xx}} + (\xi'' h_y + 2 \xi' h_{xy}) \frac{\partial}{\partial h_{xy}} \\
+ \xi' h_{yy} \frac{\partial}{\partial h_{yy}} + (\xi''' k + 3 \xi'' k_x + 3 \xi' k_{xx}) \frac{\partial}{\partial k_{xx}} + (\xi'' k_y + 2 \xi' k_{xy}) \frac{\partial}{\partial k_{xy}} + \xi' k_{yy} \frac{\partial}{\partial k_{yy}}. \]

Since the function \( \xi(x) \) is arbitrary, its derivatives \( \xi'(x), \xi''(x), \xi'''(x) \) can be treated as new arbitrary functions. Consequently, singling out in the above operator the terms with different derivatives of \( \xi(x) \), one obtains the following independent generators:

\[ X_{\xi} = \frac{\partial}{\partial x}, \quad X_{\xi''} = h \frac{\partial}{\partial h_{xx}} + k \frac{\partial}{\partial k_{xx}}, \quad (8.23) \]

\[ X_{\xi'} = h \frac{\partial}{\partial h} + k \frac{\partial}{\partial k} + 2h_x \frac{\partial}{\partial h_x} + h_y \frac{\partial}{\partial h_y} + 2k_x \frac{\partial}{\partial k_x} + k_y \frac{\partial}{\partial k_y} + 3h_{xx} \frac{\partial}{\partial h_{xx}} \]
\[ + 2h_{xy} \frac{\partial}{\partial h_{xy}} + h_{yy} \frac{\partial}{\partial h_{yy}} + 3k_{xx} \frac{\partial}{\partial k_{xx}} + 2k_{xy} \frac{\partial}{\partial k_{xy}} + k_{yy} \frac{\partial}{\partial k_{yy}}, \]

\[ X_{\xi''} = h \frac{\partial}{\partial h_x} + k \frac{\partial}{\partial k_x} + 3h_x \frac{\partial}{\partial h_{xx}} + h_y \frac{\partial}{\partial h_{xy}} + 3k_x \frac{\partial}{\partial k_{xx}} + k_y \frac{\partial}{\partial k_{xy}}. \]

Likewise, the infinitesimal transformation (8.10) of the variable \( y \),

\[ y \approx y + \varepsilon \eta(y) \quad (8.24) \]
provides the following generator:

\[ Y = -\eta(y) \frac{\partial}{\partial y} + \eta'(y) \left[ h \frac{\partial}{\partial h} + k \frac{\partial}{\partial k} \right]. \quad (8.25) \]

Its second prolongation gives rise to the following independent generators:

\[ Y_{\eta} = \frac{\partial}{\partial y}, \quad Y_{\eta''} = h \frac{\partial}{\partial h_{yy}} + k \frac{\partial}{\partial k_{yy}}, \quad (8.26) \]

\[ Y_{\eta'} = h \frac{\partial}{\partial h} + k \frac{\partial}{\partial k} + h_x \frac{\partial}{\partial h_x} + 2h_y \frac{\partial}{\partial h_y} + k_x \frac{\partial}{\partial k_x} + 2k_y \frac{\partial}{\partial k_y} + h_{xx} \frac{\partial}{\partial h_{xx}} \]
\[ + 2h_{xy} \frac{\partial}{\partial h_{xy}} + 3h_{yy} \frac{\partial}{\partial h_{yy}} + k_{xx} \frac{\partial}{\partial k_{xx}} + 2k_{xy} \frac{\partial}{\partial k_{xy}} + 3k_{yy} \frac{\partial}{\partial k_{yy}}, \]

\[ Y_{\eta''} = h \frac{\partial}{\partial h_y} + k \frac{\partial}{\partial k_y} + h_x \frac{\partial}{\partial h_{xy}} + 3h_y \frac{\partial}{\partial h_{yy}} + k_x \frac{\partial}{\partial k_{xy}} + 3k_y \frac{\partial}{\partial k_{yy}}. \]
It follows from the invariance condition under the translations, i.e. from the equations $X_\xi(J) = 0$ and $Y_\eta(J) = 0$, that $J$ in (8.18) does not depend upon $x$ and $y$. It is also evident from (8.26), that the equations

$$h = 0, \quad k = 0$$

(8.27)

are invariant under the operators (8.26). In what follows, we assume that the Laplace invariants do not vanish simultaneously, e.g. $h \neq 0$. The equation $X_\xi J = 0$ for $J = J(h, k)$ gives one of Ovsyannikov’s invariants [7], namely

$$p = \frac{k}{h}.$$  

(8.28)

One can readily verify by inspection that $p$ satisfies the invariance test for all operators (8.23) and (8.26). Moreover, the equations $X_\xi'' J = 0$ and $Y_\eta'' J = 0$ show that $h_{xx}, h_{yy}, k_{xx},$ and $k_{yy}$ can appear only in the combinations

$$r = k_{xx} - p h_{xx}, \quad s = k_{yy} - p h_{yy}.$$  

(8.29)

Thus, the general form (8.18) for the second-order invariants is reduced to

$$J(h, p, h_x, h_y, k_x, k_y, r, s).$$  

(8.30)

One has to subject the function (8.30) to the invariance test

$$X_\xi'(J) = 0, \quad X_\xi''(J) = 0, \quad Y_\eta'(J) = 0, \quad Y_\eta''(J) = 0,$$

(8.31)

where the operators $X_\xi', X_\xi'', Y_\eta'$, and $Y_\eta''$ are rewritten in terms of the variables involved in (8.30) and have the form:

$$X_\xi' = h \frac{\partial}{\partial h} + 2h_x \frac{\partial}{\partial h_x} + 2h_y \frac{\partial}{\partial h_y} + 2k_x \frac{\partial}{\partial k_x} + k_y \frac{\partial}{\partial k_y} + 2h_{xy} \frac{\partial}{\partial h_{xy}} + 2k_{xy} \frac{\partial}{\partial k_{xy}} + 3r \frac{\partial}{\partial r} + s \frac{\partial}{\partial s},$$

$$X_\xi'' = h \frac{\partial}{\partial h_x} + ph \frac{\partial}{\partial k_x} + h_x \frac{\partial}{\partial h_x} + 2h_y \frac{\partial}{\partial h_y} + k_y \frac{\partial}{\partial k_y} + 3(k_x - ph_x) \frac{\partial}{\partial r},$$

$$Y_\eta' = h \frac{\partial}{\partial h} + h_x \frac{\partial}{\partial h_x} + 2h_y \frac{\partial}{\partial h_y} + k_x \frac{\partial}{\partial k_x} + 2k_y \frac{\partial}{\partial k_y} + 2h_{xy} \frac{\partial}{\partial h_{xy}} + 2k_{xy} \frac{\partial}{\partial k_{xy}} + r \frac{\partial}{\partial r} + 3s \frac{\partial}{\partial s},$$

$$Y_\eta'' = h \frac{\partial}{\partial h_y} + ph \frac{\partial}{\partial k_y} + h_x \frac{\partial}{\partial h_x} + 2h_y \frac{\partial}{\partial h_y} + k_x \frac{\partial}{\partial k_x} + 2k_y \frac{\partial}{\partial k_y} + 3(k_y - ph_y) \frac{\partial}{\partial s}. $$  

(8.32)
The operators (8.32) obey the commutator relations

\[
[X_{\xi'}, X_{\xi''}] = -X_{\xi''}, \quad [X_{\xi'}, Y_{\eta'}] = 0, \quad [X_{\xi''}, Y_{\eta'}] = 0,
\]

\[
[X_{\xi''}, Y_{\eta''}] = 0, \quad [X_{\xi''}, Y_{\eta''}] = 0, \quad [Y_{\eta''}, Y_{\eta''}] = -Y_{\eta''},
\]

and hence span a four-dimensional Lie algebra.

According to the above table of commutators, it is advantageous to begin the solutions of the system (8.31) with the equations (see [22], Section 4.5.3)

\[
X_{\xi''}(J) = h \frac{\partial J}{\partial h_x} + p h \frac{\partial J}{\partial k_x} + h_y \frac{\partial J}{\partial h_{xy}} + k_y \frac{\partial J}{\partial k_{xy}} + 3(k_x - ph_x) \frac{\partial J}{\partial r} = 0,
\]

\[
Y_{\eta''}(J) = h \frac{\partial J}{\partial h_y} + p h \frac{\partial J}{\partial k_y} + h_x \frac{\partial J}{\partial h_{xy}} + k_x \frac{\partial J}{\partial k_{xy}} + 3(k_y - ph_y) \frac{\partial J}{\partial s} = 0.
\]

Integration of the characteristic system for the first equation:

\[
\frac{dh_x}{h} = \frac{dk_x}{ph} = \frac{dh_{xy}}{h_y} = \frac{dk_{xy}}{k_y} = \frac{dr}{3(k_x - ph_x)}
\]

yields that \( J \) involves the variables \( h, p, h_y, k_y, s \) and the following combinations:

\[
\lambda = k_x - ph_x, \quad \tau = hh_{xy} - h_x h_y, \quad \nu = ph k_{xy} - k_x k_y, \quad \omega = hr - 3\lambda h_x.
\]

Then the second equation reduces to the form

\[
Y_{\eta''}(J) = h \frac{\partial J}{\partial h_y} + p h \frac{\partial J}{\partial k_y} + 3(h_y - ph_y) \frac{\partial J}{\partial s} = 0.
\]

Integration of this equation shows that \( J = J(h, p, \lambda, \mu, \tau, \nu, \omega, \rho) \), where

\[
\lambda = k_x - ph_x, \quad \mu = k_y - ph_y, \quad \tau = hh_{xy} - h_x h_y,
\]

\[
\nu = ph k_{xy} - k_x k_y, \quad \omega = hr - 3\lambda h_x, \quad \rho = hs - 3\mu h_y.
\]

(8.33)

We solve the equation \((X_{\xi'} - Y_{\eta'})(J) = 0\) written in the variables \( h, p, \lambda, \mu, \tau, \nu, \omega, \rho \):

\[
(X_{\xi'} - Y_{\eta'})(J) = \lambda \frac{\partial J}{\partial \lambda} - \mu \frac{\partial J}{\partial \mu} + 2\omega \frac{\partial J}{\partial \omega} - 2\rho \frac{\partial J}{\partial \rho} = 0,
\]

and see that \( J = J(h, p, m, \tau, \nu, n, N) \), where

\[
m = \lambda \mu, \quad n = \omega \rho, \quad N = \frac{\omega}{\lambda}.
\]

(8.34)

To complete integration of the system (8.31) we solve the equation \( X_{\xi'}(J) = 0 \):

\[
X_{\xi'}(J) = h \frac{\partial J}{\partial h} + 3\tau \frac{\partial J}{\partial \tau} + 3\nu \frac{\partial J}{\partial \nu} + 3m \frac{\partial J}{\partial m} + 6n \frac{\partial J}{\partial n} = 0,
\]
and obtain the following six independent second-order invariants:

\[ p = \frac{k}{h}, \quad q = \frac{\tau}{h^3}, \quad Q = \frac{\nu}{h^3}, \quad N = \frac{\omega}{\lambda^2}, \quad M = \frac{n}{h^6}, \quad I = \frac{m}{h^3}, \tag{8.35} \]

provided that \( h \neq 0 \) and \( \lambda \neq 0 \). Note, that each of the equations

\[ \lambda \equiv k_x - ph_x = 0, \quad \mu \equiv k_y - ph_y = 0 \tag{8.36} \]

is invariant. We exclude this case, as well as (8.27), in our calculations.

Let us rewrite the invariants (8.35) in terms of Laplace’s semi-invariants \( h \) and \( k \), and Ovsyannikov’s invariant \( p = k/h \). Using the equations

\[ k_x - ph_x \equiv \frac{h k_x - k h_x}{h} = h p_x, \quad k_y - ph_y \equiv \frac{h k_y - k h_y}{h} = h p_y, \]

we have:

\[ \lambda = k_x - ph_x = h p_x, \quad \mu = k_y - ph_y = h p_y, \]

\[ r = k_{xx} - ph_{xx} = h p_{xx} + 2h_x p_x, \quad \omega = h^2 p_{xx} - h x p_x, \quad s = k_{yy} - ph_{yy} = h p_{yy} + 2h_y p_y, \quad \rho = h^2 p_{yy} - h y p_y \tag{8.37} \]

Using these equations, one can easily see that

\[ q = \frac{\tau}{h^3} = \frac{h_{xy}}{h^2} - \frac{h_x h_y}{h^3} \equiv \frac{1}{h} \frac{\partial^2 \ln |h|}{\partial x \partial y}, \tag{8.38} \]

and that \( Q = p^3 \tilde{q} \), where \( \tilde{q} \) is an invariant (since \( p^3 \) is an invariant) defined by

\[ \tilde{q} = \frac{1}{k} \frac{\partial^2 \ln |k|}{\partial x \partial y}. \tag{8.39} \]

Furthermore, we can replace the invariant

\[ M = \frac{\omega}{h^6} = \left( \frac{p_x}{h} \right)_x \left( \frac{p_y}{h} \right)_y \tag{8.40} \]

by the invariant

\[ H = \frac{\rho}{\mu^2}, \tag{8.41} \]

using the equations (8.34)-(8.37) and noting that \( M = N H T^2 \). Indeed,

\[ NH = \frac{\omega \rho}{\lambda^2 \mu^2} = \frac{\omega \rho}{h^4 p_x^2 p_y^2} = \frac{\omega \rho}{h^6 I^2}. \]
Invoking the definitions (8.33)-(8.35) and the equations (8.37), we have:

\[ N = \frac{\omega}{\lambda^2} = \frac{h(k_{xx} - ph_{xx})}{(k_x - ph_x)^2} - \frac{3h_x}{k_x - ph_x} = \frac{p_{xx}}{p_x^2} - \frac{h_x}{hp_x} = \frac{1}{p_x} \left( \ln \left| \frac{p_x}{h} \right| \right)_x. \]  

(8.42)

Likewise, we rewrite the invariant (8.41) in the form

\[ H = \frac{\rho}{\mu^2} = \frac{p_{yy}}{p_y^2} - \frac{h_y}{hp_y} = \frac{1}{p_y} \left( \ln \left| \frac{p_y}{h} \right| \right)_y. \]  

(8.43)

Finally, we have

\[ I = \frac{\lambda \mu}{h^3} = \frac{p_x p_y}{h}. \]  

(8.44)

Collecting together the invariants (8.28), (8.38), (8.39), (8.42), (8.43) and (8.44), we ultimately arrive at the following complete set of invariants of the second order for equation (8.1):

\[ p = \frac{k}{h}, \quad q = \frac{1}{h} \frac{\partial^2 \ln |h|}{\partial x \partial y}, \quad \tilde{q} = \frac{1}{k} \frac{\partial^2 \ln |k|}{\partial x \partial y}, \]  

(8.45)

\[ N = \frac{1}{p_x} \frac{\partial}{\partial x} \ln \left| \frac{p_x}{h} \right|, \quad H = \frac{1}{p_y} \frac{\partial}{\partial y} \ln \left| \frac{p_y}{h} \right|, \quad I = \frac{p_x p_y}{h}. \]  

(8.46)

Besides, we have four individually invariant equations (8.27) and (8.36):

\[ h = 0, \quad k = 0, \quad k_x - ph_x = 0, \quad k_y - ph_y = 0. \]  

(8.47)

### 8.4 Invariant differentiation and a basis of invariants. Solution of Laplace’s problem

Let us find the invariant differentiations converting any invariant of equation (8.1) into invariants of the same equation. Recall that given any group with generators

\[ X_\nu = \xi_\nu(x, u) \frac{\partial}{\partial x^i} + \eta_\nu(x, u) \frac{\partial}{\partial u^\alpha}, \]

where \( x = (x^1, \ldots, x^n) \) are \( n \) independent variables, there exist \( n \) invariant differentiations of the form (see [25], Chapter 7, or [22], Section 8.3.5)

\[ D = f^i D_i \]  

(8.48)

where the coefficients \( f^i(x, u, u(1), u(2), \ldots) \) are defined by the equations

\[ X_\nu(f^i) = f^j D_j(\xi^i_\nu), \quad i = 1, \ldots, n. \]  

(8.49)
In our case, the generators \( X_\nu \) are replaced by the operators (8.22), (8.25). Let us write the invariant differential operator (8.48) in the form

\[
\mathcal{D} = f D_x + g D_y. 
\]  
(8.50)

The equations (8.49) for the coefficients become:

\[
\begin{align*}
X(f) &= f D_x(\xi(x)) + g D_y(\xi(x)) \equiv -\xi'(x)f, \quad X(g) = 0; \\
Y(g) &= f D_x(\eta(y)) + g D_y(\eta(y)) \equiv -\eta'(y)g, \quad Y(f) = 0. 
\end{align*} 
\]  
(8.51)

Here \( f, g \) are unknown functions of \( x, y, h, k, h_x, h_y, k_x, k_y, h_{xx}, \ldots \). The operators \( X \) and \( Y \) are prolonged to all derivatives of \( h, k \) involved here.

Let us begin with \( f = f(x, y, h, k) \) and \( g = g(x, y, h, k) \). Then equations (8.51) give the following equations for \( f \):

\[
\begin{align*}
\xi \frac{\partial f}{\partial x} - \xi' \left[ h \frac{\partial f}{\partial h} + k \frac{\partial f}{\partial k} \right] &= \xi'(x)f, \\
\eta \frac{\partial f}{\partial y} - \eta' \left[ h \frac{\partial f}{\partial h} + k \frac{\partial f}{\partial k} \right] &= 0.
\end{align*}
\]

Invoking that \( \xi, \xi', \eta \) and \( \eta' \) are arbitrary functions (cf. the previous section), we obtain the following four equations:

\[
\frac{\partial f}{\partial x} = 0, \quad h \frac{\partial f}{\partial h} + k \frac{\partial f}{\partial k} = -f, \quad \frac{\partial f}{\partial y} = 0, \quad h \frac{\partial f}{\partial h} + k \frac{\partial f}{\partial k} = 0,
\]

whence \( f = 0 \). Likewise, the equations (8.51) written for \( g = g(x, y, h, k) \) yield \( g = 0 \). Thus, there are no invariant differentiations (8.50) with \( f = f(x, y, h, k) \) and \( g = g(x, y, h, k) \).

Therefore, we continue our search by letting

\[
f = f(x, y, h, k, h_x, h_y, k_x, k_y), \quad g = g(x, y, h, k, h_x, h_y, k_x, k_y).
\]

The first-order prolongations of the generators \( X \) and \( Y \) furnish the operators (cf. (8.23) and (8.26))

\[
\begin{align*}
&X_\xi = \frac{\partial}{\partial x}, \quad X_{\xi''} = h \frac{\partial}{\partial h_x} + k \frac{\partial}{\partial k_x}, \\
&X_{\xi'} = h \frac{\partial}{\partial h} + k \frac{\partial}{\partial k} + 2h_x \frac{\partial}{\partial h_x} + 2h_y \frac{\partial}{\partial h_y} + 2k_x \frac{\partial}{\partial k_x} + 2k_y \frac{\partial}{\partial k_y}, \tag{8.52}
\end{align*}
\]

and

\[
\begin{align*}
&Y_\eta = \frac{\partial}{\partial y}, \quad Y_{\eta''} = h \frac{\partial}{\partial h_y} + k \frac{\partial}{\partial k_y}, \\
&Y_{\eta'} = h \frac{\partial}{\partial h} + k \frac{\partial}{\partial k} + h_x \frac{\partial}{\partial h_x} + 2h_y \frac{\partial}{\partial h_y} + k_x \frac{\partial}{\partial k_x} + 2k_y \frac{\partial}{\partial k_y}, \tag{8.53}
\end{align*}
\]
respectively. The operators $X_\xi$ and $X_\eta$ yield that the functions $f, g$ do not involve the variables $x$ and $y$. Furthermore, equations (8.51) split into the equations

$$X_\xi'(f) = -f, \quad X_\eta'(f) = 0, \quad Y_\eta'(f) = 0, \quad Y_\eta''(f) = 0$$

(8.54)

and

$$X_\xi'(g) = 0, \quad X_\eta'(g) = 0, \quad Y_\eta'(g) = -g, \quad Y_\eta''(g) = 0$$

(8.55)

for functions $f(h, k, h_x, h_y, k_x, k_y)$ and $g(h, k, h_x, h_y, k_x, k_y)$, respectively. The equations $X_\xi'(f) = 0, Y_\eta'(f) = 0$ and $X_\xi'(g) = 0, Y_\eta'(g) = 0$ yield that $f$ and $g$ depend on the following four variables (cf. the previous section):

$$h, \quad k, \quad \lambda = k_x - ph_x = hp_x, \quad \mu = k_y - ph_y = hp_y.$$  

We rewrite the operators $X_\xi'$ and $Y_\eta'$ in the variables $h, \lambda, \mu$ and $p = k/h$:

$$X_\xi' = h \frac{\partial}{\partial h} + 2\lambda \frac{\partial}{\partial \lambda} + \mu \frac{\partial}{\partial \mu}, \quad Y_\eta' = h \frac{\partial}{\partial h} + \lambda \frac{\partial}{\partial \lambda} + 2\mu \frac{\partial}{\partial \mu},$$

(8.56)

integrate the equations

$$X_\xi'(f) = -f, \quad Y_\eta'(f) = 0$$

and

$$X_\xi'(g) = 0, \quad Y_\eta'(g) = -g$$

for the functions $f(h, p, \lambda, \mu)$ and $g(h, p, \lambda, \mu)$, respectively, and obtain:

$$f = \frac{h}{\lambda} F(p, I), \quad g = \frac{h}{\mu} G(p, I),$$

(8.57)

where $p$ and $I$ are the invariants (8.28) and (8.44), respectively:

$$p = \frac{k}{h}, \quad I = \frac{\lambda \mu}{h^3} = \frac{p_x p_y}{h}.$$  

Substituting the expressions (8.57) in (8.50), one obtains the invariant differentiation

$$\mathcal{D} = F(p, I) \frac{1}{p_x} D_x + G(p, I) \frac{1}{p_y} D_y$$

(8.58)

with arbitrary functions $F(p, I)$ and $G(p, I)$.

**Remark 8.2.** The most general invariant differentiation has the form (8.58) where $F(p, I)$ and $G(p, I)$ are replaced by arbitrary functions of higher-order invariants, e.g. by $F(p, I, q, \tilde{q}, N, H)$ and $G(p, I, q, \tilde{q}, N, H)$, provided that the corresponding invariants are known. It suffices, however, to let that $F$ and $G$ are any constants.
Letting in (8.58) $F = 1$, $G = 0$ and then $F = 0$, $G = 1$, one obtains the following simplest invariant differentiations in directions $x$ and $y$, respectively:

\[
\mathcal{D}_1 = \frac{1}{p_x} D_x, \quad \mathcal{D}_2 = \frac{1}{p_y} D_y.
\] (8.59)

Now, one can construct higher-order invariants by means of the invariant differentiations (8.59) and prove the following statement.

**Theorem 8.3.** A basis of invariants of an arbitrary order for equation (8.1) is provided by the invariants

\[
p = \frac{k}{h}, \quad I = \frac{p_x p_y}{h}, \quad q = \frac{1}{h} \frac{\partial^2 \ln |h|}{\partial x \partial y}, \quad \tilde{q} = \frac{1}{k} \frac{\partial^2 \ln |k|}{\partial x \partial y}
\] (8.60)

or, alternatively, by the invariants

\[
p = \frac{k}{h}, \quad I = \frac{p_x p_y}{h}, \quad N = \frac{1}{p_x} \frac{\partial}{\partial x} \ln \left| \frac{p_x}{h} \right|, \quad q = \frac{1}{h} \frac{\partial^2 \ln |h|}{\partial x \partial y}.
\] (8.61)

**Proof.** The reckoning shows that the operators act as follows:

\[
\mathcal{D}_1(p) = 1, \quad \mathcal{D}_1(I) = \left( N + \frac{1}{p} \right) I + p(p\tilde{q} - q),
\]

\[
\mathcal{D}_2(p) = 1, \quad \mathcal{D}_2(I) = \left( H + \frac{1}{p} \right) I + p(p\tilde{q} - q).
\]

Hence, the invariants (8.61) can be obtained from (8.60) by invariant differentiations, and vice versa. Consequently, a basis of all invariants of the second order (8.45)-(8.46) is provided by (8.60) or by (8.61). Furthermore, one can show, invoking equations (8.37), that the invariant differentiations $\mathcal{D}_1$ and $\mathcal{D}_2$ of the basic invariants (8.60), or (8.61) provide 6 independent invariants involving third-order partial derivatives of $h$ and $k$. On the other hand, consideration of third-order invariants involves 8 third-order derivatives of $h$ and $k$. However, the invariance condition brings two additional equations due to the fourth-order derivatives $\xi^{(iv)}(x)$ and $\eta^{(iv)}(y)$, so that we will have precisely 6 additional invariants, just as given by invariant differentiations. The same reasoning for higher-order derivatives completes the proof.

### 8.5 Representation of invariants in alternative coordinates

It can be useful to write the invariants of the hyperbolic equations in the coordinates

\[
z = x + y, \quad t = x - y.
\] (8.62)
We have:

\[ x = \frac{z + t}{2}, \quad y = \frac{z - t}{2}, \]
\[ u_x = u_z + u_t, \quad u_y = u_z - u_t, \quad u_{xy} = u_{zz} - u_{tt}. \]

Then Eq. (8.1) is written in the alternative standard form:

\[ u_{zz} - u_{tt} + \tilde{a}(z, t)u_z + \tilde{b}(z, t)u_t + \tilde{c}(z, t)u = 0, \quad (8.63) \]

where

\[ \tilde{a}(z, t) = a(x, y) + b(x, y), \]
\[ \tilde{b}(z, t) = a(x, y) - b(x, y), \quad (8.64) \]
\[ \tilde{c}(z, t) = c(x, y). \]

The equivalence algebra \( L_E \) for Eq. (8.63) can be obtained without solving again the determining equations. Rather we obtain it by rewriting the generator (8.3) in the alternative coordinates (8.62). Let us consider, for the sake of brevity, the \((x, y, u)\) part of the operator (8.3) and write it as follows:

\[ Y = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \sigma u \frac{\partial}{\partial u}, \quad \xi_y = 0, \quad \eta_x = 0. \]

Rewriting it in the new variables (8.62), we have:

\[ Y = \tilde{\xi} \frac{\partial}{\partial z} + \tilde{\eta} \frac{\partial}{\partial t} + \tilde{\sigma} u \frac{\partial}{\partial u}, \]

where

\[ \tilde{\xi} = \xi + \eta, \quad \tilde{\eta} = \xi - \eta, \quad \tilde{\sigma} = \sigma. \]

Now we rewrite the conditions \( \xi_y = 0, \eta_x = 0 \) in terms of \( \tilde{\xi}(z, t), \tilde{\eta}(z, t) \). We have

\[ \xi = \frac{1}{2}(\tilde{\xi} + \tilde{\eta}), \quad \eta = \frac{1}{2}(\tilde{\xi} - \tilde{\eta}) \]

and

\[ \xi_y = \frac{1}{2}(\tilde{\xi}_z + \tilde{\eta}_z - \tilde{\xi}_t - \tilde{\eta}_t), \quad \eta_x = \frac{1}{2}(\tilde{\xi}_z - \tilde{\eta}_z + \tilde{\xi}_t - \tilde{\eta}_t). \]

Hence, the conditions \( \xi_y = 0, \eta_x = 0 \) become:

\[ \tilde{\xi}_z + \tilde{\eta}_z - \tilde{\xi}_t - \tilde{\eta}_t = 0, \quad \tilde{\xi}_z - \tilde{\eta}_z + \tilde{\xi}_t - \tilde{\eta}_t = 0, \]

whence

\[ \tilde{\xi}_z = \tilde{\eta}_t, \quad \tilde{\eta}_z = \tilde{\xi}_t. \]

This proves the following theorem.
Theorem 8.4. The equivalence algebra $L_E$ for the family of equations (8.63) comprises the operators (properly extended to the coefficients $\tilde{a}, \tilde{b}, \tilde{c}$)

$$Y = \tilde{\xi}(z, t) \frac{\partial}{\partial z} + \tilde{\eta}(z, t) \frac{\partial}{\partial t} + \tilde{\sigma}(z, t) u \frac{\partial}{\partial u},$$

(8.65)

where $\tilde{\beta}(z, t)$ is an arbitrary function, whereas $\tilde{\xi}(z, t), \tilde{\eta}(z, t)$ solve the equations

$$\tilde{\xi}_z = \tilde{\eta}_t, \quad \tilde{\eta}_z = \tilde{\xi}_t.$$

(8.66)

The operator (8.65) generates the continuous infinite group $\mathcal{E}_c$ of equivalence transformations composed of the linear transformation of the dependent variable:

$$u = \psi(z, t) v, \quad \psi(x, y) \neq 0,$$

and the following change of the independent variables:

$$z = f \left( \frac{z + t}{2} \right) + g \left( \frac{z - t}{2} \right), \quad t = f \left( \frac{z + t}{2} \right) - g \left( \frac{z - t}{2} \right).$$

The invariants of Eq. (8.63) are readily obtained by applying the transformation (8.62) to the invariants invariants of the previous section. Namely, we have from (8.64):

$$a = \frac{\tilde{\alpha} + \tilde{b}}{2}, \quad b = \frac{\tilde{\alpha} - \tilde{b}}{2}, \quad c = \tilde{c},$$

and hence, invoking (8.62):

$$a_x = \frac{1}{2} \left( \tilde{a}_z + \tilde{a}_t + \tilde{b}_z + \tilde{b}_t \right), \quad b_y = \frac{1}{2} \left( \tilde{a}_z - \tilde{a}_t - \tilde{b}_z + \tilde{b}_t \right).$$

The Laplace invariants (8.17) become

$$h = \frac{1}{2} \left( \tilde{a}_z + \tilde{a}_t + \tilde{b}_z + \tilde{b}_t \right) + \frac{1}{4} \left( \tilde{a}^2 - \tilde{b}^2 \right) - \tilde{c},$$

$$k = \frac{1}{2} \left( \tilde{a}_z - \tilde{a}_t - \tilde{b}_z + \tilde{b}_t \right) + \frac{1}{4} \left( \tilde{a}^2 - \tilde{b}^2 \right) - \tilde{c}.$$ 

It is convenient to take their linear combinations $h + k, h - k$ and obtain the following semi-invariants for Eq. (8.62):

$$\tilde{h} = \tilde{a}_z + \tilde{b}_t + \frac{1}{2} \left( \tilde{a}^2 - \tilde{b}^2 \right) - 2\tilde{c}, \quad \tilde{k} = \tilde{a}_t + \tilde{b}_z.$$

(8.67)

We have

$$h = \frac{1}{2} (\tilde{h} + \tilde{k}), \quad k = \frac{1}{2} (\tilde{h} - \tilde{k}),$$

(8.68)
and Ovsyannikov’s invariant \( p \) (see (8.60)) is written in the form

\[
p = \frac{k}{\overline{h}} = \frac{\overline{h} - \overline{k}}{\overline{h} + \overline{k}} \equiv \frac{1 - (\overline{k}/\overline{h})}{1 + (\overline{k}/\overline{h})}.
\] (8.69)

It follows that

\[
\frac{\overline{k}}{\overline{h}} = \frac{1 - p}{1 + p}.
\]

Thus, we get the following Ovsyannikov’s invariant for Eq. (8.63):

\[
\overline{p} = \frac{\overline{k}}{\overline{h}}.
\] (8.70)

Likewise, we can readily rewrite the invariant \( I \) from (8.60) in the alternative coordinates and obtain an invariant for Eq. (8.63). Invoking (8.68), we have:

\[
I = \frac{p_x p_y}{h} = \frac{2}{h + k} \left(p_z + p_t\right) \left(p_z - p_t\right) = \frac{2}{h + k} \left(p_z^2 - p_t^2\right).
\] (8.71)

Now we get, use the expression (8.68) for \( p \):

\[
p_z = 2 \frac{\overline{k}h_z - \overline{h}k_z}{(h + k)^2}, \quad p_t = 2 \frac{\overline{k}h_t - \overline{h}k_t}{(h + k)^2},
\] (8.72)

and substitute in (8.71) to obtain the following expression for the invariant (8.71):

\[
I = \frac{8}{(h + k)^5} \left[(\overline{k}h_z - \overline{h}k_z)^2 - (\overline{k}h_t - \overline{h}k_t)^2\right].
\]

Since \( \overline{h} + \overline{k} = (\overline{p} + 1)\overline{k} \) and \( \overline{p} \) is an invariant (see (8.70)), we can replace \( \overline{h} + \overline{k} \) by \( \overline{k} \) and, ignoring the constant coefficient, obtain the following invariant for Eq. (8.63):

\[
\tilde{I} = \frac{1}{\overline{k}^5} \left[(\overline{k}h_z - \overline{h}k_z)^2 - (\overline{k}h_t - \overline{h}k_t)^2\right].
\] (8.73)

Furthermore, we can readily obtain from (8.59) the corresponding invariant differentiations for Eq. (8.63). We have:

\[
\mathcal{D}_1 = \frac{1}{p_x} \frac{D_x}{D_z} = \frac{1}{p_x} \left(D_z + D_t\right), \quad \mathcal{D}_2 = \frac{1}{p_y} \frac{D_y}{D_z} = \frac{1}{p_y} \left(D_z - D_t\right).
\]

Substituting here the following expressions for \( p_x \) and \( p_y \) (see (8.72)):

\[
p_x = p_z + p_t = \frac{2}{(h + k)^2} \left[\overline{k}(\overline{h}_z + \overline{h}_t) - \overline{h}(\overline{k}_z + \overline{k}_t)\right]
\]
\[ p_y = p_z - p_t = \frac{2}{(\tilde{h} + \tilde{k})^2} \left[ \tilde{k}(\tilde{h}_z - \tilde{h}_t) - \tilde{h}(\tilde{k}_z - \tilde{k}_t) \right], \]

and ignoring the coefficient 2, we obtain:

\[ D_1 = \frac{(\tilde{h} + \tilde{k})^2}{k(\tilde{h}_z + \tilde{h}_t) - \tilde{h}(\tilde{k}_z + \tilde{k}_t)} \left( D_z + D_t \right), \]

\[ D_2 = \frac{(\tilde{h} + \tilde{k})^2}{k(\tilde{h}_z - \tilde{h}_t) - \tilde{h}(\tilde{k}_z - \tilde{k}_t)} \left( D_z - D_t \right). \]

Since \( \tilde{h} = \tilde{p} \tilde{k} \) (see (8.70)), and hence \( \tilde{h} + \tilde{k} = (1 + \tilde{p})\tilde{k} \), the invariant differentiations for Eq. (8.63) can be written as follows:

\[ \tilde{D}_1 = \frac{\tilde{k}}{\tilde{h}_z + \tilde{h}_t - \tilde{p}(\tilde{k}_z + \tilde{k}_t)} \left( D_z + D_t \right), \]

\[ \tilde{D}_2 = \frac{\tilde{k}}{\tilde{h}_z - \tilde{h}_t - \tilde{p}(\tilde{k}_z - \tilde{k}_t)} \left( D_z - D_t \right). \] (8.74)

### 9 Invariants of elliptic equations

E. Cotton [35] extended the Laplace invariants to the elliptic equations

\[ u_{\alpha\alpha} + u_{\beta\beta} + A(\alpha, \beta)u_\alpha + B(\alpha, \beta)u_\beta + C(\alpha, \beta)u = 0 \] (9.1)

and obtained the following semi-invariants:

\[ H = A_\alpha + B_\beta + \frac{1}{2} \left( A^2 + B^2 \right) - 2C, \quad K = A_\beta - B_\alpha. \] (9.2)

Cotton’s invariants can be derived by considering the linear transformation (8.9) and proceeding as in Section 8.2. This way was illustrated in [34] where I also mentioned that Cotton’s invariants (9.2) can be obtained from the Laplace invariants (8.17) merely by a complex change of the independent variables connecting the hyperbolic and elliptic equations. Namely, the change of variables

\[ \alpha = x + y, \quad \beta = i(y - x) \] (9.3)

maps the hyperbolic equation (8.1) into the elliptic equation (9.1). We will rather use the alternative representation of invariants given in Section 8.5. Then the variables (9.3) and (8.62) are related by

\[ \alpha = z, \quad \beta = -it. \] (9.4)
It is manifest that
\[ D_z = D_\alpha, \quad D_t = -i D_\beta \]  
(9.5)
and the hyperbolic equation (8.63) becomes the elliptic equation (9.1), where
\[ A = \tilde{a}, \quad B = -i \tilde{b}, \quad C = \tilde{c}. \]  
(9.6)

**Theorem 9.1.** The equivalence algebra \( L_E \) for the family of the elliptic equations (9.1) comprises the operators (properly extended to the coefficients \( A, B, C \))
\[ Y = \xi^1(\alpha, \beta) \frac{\partial}{\partial \alpha} + \xi^2(\alpha, \beta) \frac{\partial}{\partial \beta} + \nu(\alpha, \beta) u \frac{\partial}{\partial u}, \]  
(9.7)
where \( \nu(\alpha, \beta) \) is an arbitrary function, whereas \( \xi^1(\alpha, \beta), \xi^2(\alpha, \beta) \) solve the Cauchy-Riemann equations
\[ \xi^1_\alpha = \xi^2_\beta, \quad \xi^2_\alpha = -\xi^1_\beta. \]  
(9.8)

**Proof.** We proceed as in Theorem 8.4. Namely, we rewrite the operator (8.65) in the new variables (9.4) to obtain
\[ Y = \xi^1(\alpha, \beta) \frac{\partial}{\partial \alpha} + \xi^2(\alpha, \beta) \frac{\partial}{\partial \beta} + \nu(\alpha, \beta) u \frac{\partial}{\partial u}, \]  
where
\[ \xi^1 = \tilde{\xi}, \quad \xi^2 = -i \tilde{\eta}, \quad \nu = \tilde{\sigma}. \]

We have
\[ \tilde{\xi}_z = \xi^1_\alpha, \quad \tilde{\xi}_t = -i \xi^1_\beta, \quad \tilde{\eta}_z = i \xi^2_\alpha, \quad \tilde{\eta}_t = i \xi^2_\beta. \]

It follows that the equations (8.66) become the Cauchy-Riemann system (9.8), thus proving the theorem.

Using (9.5) and (9.6), we readily obtain the Cotton invariants (9.2) from the semi-invariants (8.67). Namely, they are related as follows:
\[ \tilde{h} = H, \quad \tilde{k} = -i K. \]  
(9.9)
Likewise we transform the invariants \( \tilde{p} \) (8.70) and \( \tilde{I} \) (8.73) for the hyperbolic equation (8.63) into the following invariants for the elliptic equation (9.1):
\[ P = \frac{K}{H} \]  
(9.10)
and
\[ J = \frac{1}{K^5} \left[ (KH_z - HK_z)^2 + (KH_t - H.K_t)^2 \right]. \]  
(9.11)
We have the relations
\[ \tilde{p} = -i P, \quad \tilde{I} = -J. \] (9.12)

It is not difficult to transform in a similar way all basic invariants (8.60) and (8.61) as well as the individually invariant equations (8.47) and obtain the invariants and the individually invariant equations, respectively, for the elliptic equation (9.1).

Furthermore, one can readily obtain from the invariant differentiations (8.74) the corresponding invariant differentiations for Eq. (9.1). Indeed, using Eqs. (9.5), (9.9) and (9.12), we rewrite the operators (8.74) as follows:
\[ \tilde{\mathcal{D}}_1 = \frac{K}{(H_\beta + PK_\beta) + i(H_\alpha + PK_\alpha)} \left( D_\alpha - iD_\beta \right), \]
\[ \tilde{\mathcal{D}}_2 = \frac{-K}{(H_\beta + PK_\beta) - i(H_\alpha + PK_\alpha)} \left( D_\alpha + iD_\beta \right), \]
or
\[ \tilde{\mathcal{D}}_1 = S \left( [(H_\beta + PK_\beta)D_\alpha - (H_\alpha + PK_\alpha)D_\beta] - i[(H_\alpha + PK_\alpha)D_\alpha + (H_\beta + PK_\beta)D_\beta] \right), \]
\[ \tilde{\mathcal{D}}_2 = -S \left( [(H_\beta + PK_\beta)D_\alpha - (H_\alpha + PK_\alpha)D_\beta] + i[(H_\alpha + PK_\alpha)D_\alpha + (H_\beta + PK_\beta)D_\beta] \right), \]
where
\[ S = \frac{K}{(H_\beta + PK_\beta)^2 + (H_\alpha + PK_\alpha)^2}. \]

Singling out the real and imaginary parts obtained by taking the linear combinations
\[ \hat{\mathcal{D}}_1 = \frac{1}{2} \left( \tilde{\mathcal{D}}_1 - \tilde{\mathcal{D}}_2 \right), \quad \hat{\mathcal{D}}_2 = \frac{i}{2} \left( \tilde{\mathcal{D}}_1 + \tilde{\mathcal{D}}_2 \right), \]
we arrive at the following invariant differentiations for the elliptic equation (9.1):
\[ \hat{\mathcal{D}}_1 = \frac{K}{(H_\beta + PK_\beta)^2 + (H_\alpha + PK_\alpha)^2} \left( (H_\beta + PK_\beta)D_\alpha - (H_\alpha + PK_\alpha)D_\beta \right), \]
\[ \hat{\mathcal{D}}_2 = \frac{K}{(H_\beta + PK_\beta)^2 + (H_\alpha + PK_\alpha)^2} \left( (H_\alpha + PK_\alpha)D_\alpha + (H_\beta + PK_\beta)D_\beta \right). \]

One can rewrite in a similar way the basic invariants (8.60) and (8.61) as well as the individually invariant equations (8.47). Subjecting these basic invariants to the invariant differentiations \( \hat{\mathcal{D}}_1 \) and \( \hat{\mathcal{D}}_2 \), one obtains all invariants of the elliptic equations.
10 Semi-invariants of parabolic equations

Consider the parabolic equations written in the canonical form:

\[ u_t + a(t, x)u_{xx} + b(t, x)u_x + c(t, x)u = 0. \]  
\[ \text{(10.1)} \]

The group of equivalence transformations of the equations (10.1) is an infinite group composed of linear transformation of the dependent variable:

\[ u = \sigma(x, y)v, \quad \sigma(x, y) \neq 0, \]  
\[ \text{(10.2)} \]

and invertible changes of the independent variables of the form:

\[ \tau = \phi(t), \quad y = \psi(t, x), \]  
\[ \text{(10.3)} \]

where \( \phi(t), \psi(t, x) \) and \( \sigma(t, x) \) are arbitrary functions. To verify that (10.2)–(10.3) are equivalence transformations one can proceed as follows. The total differentiations

\[ D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + \cdots, \quad D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + \cdots \]

are transformed by (10.3) to the operators \( D_\tau \) and \( D_y \), the latter being defined by the simultaneous equations

\[ D_t = D_t(\phi) D_\tau + D_t(\psi) D_y, \quad D_x = D_x(\psi) D_y, \]  
\[ \text{or} \]

\[ D_t = \phi'(t) D_\tau + \psi_t D_y, \quad D_x = \psi_x D_y. \]  
\[ \text{(10.4)} \]

Application of (10.4) to (10.2) yields:

\[ u_t = \sigma \phi' v_t + \sigma \psi_t v_y + \sigma v, \quad u_x = \sigma \psi_x v_y + \sigma_x v, \quad u_{xx} = \sigma \psi_x^2 v_{yy} + (\sigma \psi_{xx} + 2 \sigma_x \psi_y)v_y + \sigma_{xx} v. \]

Hence, the equation (10.1) is transferred by the transformations (10.2)–(10.3) to an equation having again the form (10.1), viz.

\[ \phi' v_t + \sigma \psi_x^2 v_{yy} + \left[ \left( \psi_{xx} + 2 \frac{\sigma_x}{\sigma} \psi_x \right) a + b \psi_x + \psi \right] v + \left[ \frac{\sigma_t}{\sigma} + \frac{\sigma_{xx}}{\sigma} a + \frac{\sigma_x}{\sigma} b + c \right] v = 0. \]  
\[ \text{(10.5)} \]

Let us find semi-invariants of the equations (10.1), i.e. its invariants under the transformation (10.2). Substituting the infinitesimal transformation \( u \approx [1 + \varepsilon \eta(x, y)] v \) into the equation (10.1) (or using (10.5) with \( \phi(t) = t, \psi(t, x) = x \)) one obtains:

\[ v_t + a v_{xx} + (b + 2\varepsilon a \eta_x) v_x + [c + \varepsilon (\eta_t + a \eta_{xx} + b \eta_x)] v = 0. \]

Thus, the coefficients of the equation (10.1) undergo the infinitesimal transformations

\[ a = a, \quad b \approx b + 2\varepsilon a \eta_x, \quad c \approx c + \varepsilon (\eta_t + a \eta_{xx} + b \eta_x), \]  
\[ \text{(10.6)} \]
that provide the generator
\[ X = 2a\eta_x \frac{\partial}{\partial b} + \left( \eta_t + a\eta_{xx} + b\eta_x \right) \frac{\partial}{\partial c}. \] (10.7)

The infinitesimal test \( XJ = 0 \) for the invariants \( J(a, b, c) \) is written
\[ 2a\eta_x \frac{\partial J}{\partial b} + \left( \eta_t + a\eta_{xx} + b\eta_x \right) \frac{\partial J}{\partial c} = 0, \]
whence \( \partial J/\partial c = 0, \partial J/\partial b = 0 \). Hence, the only independent solution is
\[ J = a. \] (10.8)

Therefore, we consider the semi-invariants involving the first-order derivatives (first-order differential invariants for the operator (10.7)), i.e. those of the form
\[ J(a, a_t, a_x; b, b_t, b_x; c, c_t, c_x). \]

The once-extended generator (10.7) is:
\[
X = 2a\eta_x \frac{\partial}{\partial b} + \left( \eta_t + a\eta_{xx} + b\eta_x \right) \frac{\partial}{\partial c} + 2 \left( a\eta_{tx} + a_t \eta_x \right) \frac{\partial}{\partial b_t} + 2 \left( a\eta_{xx} + a_x \eta_x \right) \frac{\partial}{\partial b_x} \\
+ \left( \eta_{tt} + a\eta_{txx} + a_t \eta_{xx} + b\eta_{tx} + b_t \eta_x \right) \frac{\partial}{\partial c_t} + \left( \eta_{tx} + a\eta_{xxx} + a_x \eta_{xx} + b\eta_{xx} + b_x \eta_x \right) \frac{\partial}{\partial c_x}.
\]

The equation \( XJ = 0 \), upon equating to zero the terms with \( \eta_{tx}, \eta_{xxx}, \eta_{xx}, \eta_t, \eta_{xx} \), and finally with \( \eta_x \) yields
\[ \frac{\partial J}{\partial c_t} = 0, \quad \frac{\partial J}{\partial c_x} = 0, \quad \frac{\partial J}{\partial b_t} = 0, \quad \frac{\partial J}{\partial c} = 0, \quad \frac{\partial J}{\partial b_x} = 0, \quad \frac{\partial J}{\partial b} = 0. \] (10.9)

It follows that
\[ J = J(a, a_t, a_x). \]

Thus, there are no first-order differential invariants other than the trivial ones, i.e. \( J = J(a, a_t, a_x) \). Therefore, let us look for the semi-invariants of the second order (second-order differential invariants), i.e. those of the form
\[ J(a, a_t, a_x, a_{tx}, a_{xx}; b, b_t, b_x, b_{tx}, b_{xx}; c, c_t, c_x; c_{tt}, c_{tx}, c_{xx}). \]

We take the twice-extended generator (10.7) and proceed as above. Then we first arrive at the equations
\[ \frac{\partial J}{\partial c_{tt}} = 0, \quad \frac{\partial J}{\partial c_{tx}} = 0, \quad \frac{\partial J}{\partial c_{xx}} = 0, \quad \frac{\partial J}{\partial b_{tt}} = 0, \quad \frac{\partial J}{\partial b_{tx}} = 0, \quad \frac{\partial J}{\partial c_t} = 0, \quad \frac{\partial J}{\partial c} = 0. \] (10.10)
Eqs. (10.10) yield that
\[
J = J(a, a_t, a_x, a_{tt}, a_{tx}, a_{xx}; b, b_t, b_x, b_{xx}; c_x),
\]
and the equation \(XJ = 0\) reduces to the following system of four equations:
\[
\frac{\partial J}{\partial c_x} + 2a \frac{\partial J}{\partial b_t} = 0, \quad a \frac{\partial J}{\partial b_t} - \frac{\partial J}{\partial b_{xx}} = 0, \quad a \frac{\partial J}{\partial b_x} + (a_x - b) \frac{\partial J}{\partial b_{xx}} = 0,
\]
\[
a \frac{\partial J}{\partial b} + a_t \frac{\partial J}{\partial b_t} + a_x \frac{\partial J}{\partial b_x} + (a_{xx} - b_x) \frac{\partial J}{\partial b_{xx}} = 0. \tag{10.11}
\]
Solving the system (10.11), we arrive at the following result obtained in [34].

**Theorem 10.1.** The semi-invariants of the second order for the family of parabolic equations (10.1) have the form
\[
J = \Phi(K, a, a_t, a_x, a_{tt}, a_{tx}, a_{xx}), \tag{10.12}
\]
where \(\Phi\) is an arbitrary function and
\[
K = \frac{1}{2} b^2 a_x + \left( a_t + a a_{xx} - a_x^2 \right) b + (a a_x - ab) b_x - ab_t - a^2 b_{xx} + 2a^2 c_x. \tag{10.13}
\]
Hence, the quantity \(K\) given by (10.13) is the main semi-invariant and furnishes, together with \(a\), a basis of the second-order semi-invariants.

**Remark 10.1.** In addition to the semi-invariants (10.12), there is an invariant equation with respect to the equivalence transformations (10.2)–(10.3). This singular invariant equations is derived in [36] and involves the derivatives of the coefficient \(a\) up to fifth-order and the derivatives of \(K\) with respect to \(x\) up to second order.

### 11 Invariants of non-linear wave equations

#### 11.1 The equations \(v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)\)

In [37], the second-order invariants are obtained for Eqs. (4.1),
\[
v_{tt} = f(x, v_x)v_{xx} + g(x, v_x).
\]
The following invariants with respect to the infinite equivalence algebra (4.11) provides a basis of invariants:

\[ \lambda = \frac{ff_{22}}{(f_2)^2}, \]

\[ \mu = \frac{ff_{22}(2g_2 - f_1) - ff_{12} - 3(f_2)^2(g_2 - f_1)}{f_2 [f_2(g_2 - f_1) + f(f_{12} - g_{22})]} , \]

\[ \nu = \frac{f_1(f_1f_{22} + 2f_2g_{22}) + 4g_2[f_{22}(g_2 - f_1) - f_2g_{22}]}{[f_2(g_2 - f_1) + f(f_{12} - g_{22})]^2} \]

\[ - (f_2)^2 \frac{2[(f_1)^2 + (g_2)^2] + f(f_{11} - 2g_{12}) + f_{21}g_1 - 5f_1g_2}{[f_2(g_2 - f_1) + f(f_{12} - g_{22})]^2} \]

Here the subscripts denote the respective differentiations:

\[ f_1 = \frac{\partial f}{\partial x}, \quad f_2 = \frac{\partial f}{\partial v_x}, \ldots . \]

Furthermore, the following four individually invariant equations are singled out:

\[ f_2 = \frac{\partial f}{\partial v_x} = 0, \]

\[ f_2 (g_2 - f_1) + f(f_{12} - g_{22}) = 0, \]

\[ ff_{22}(2g_2 - f_1) - ff_{12} - 3(f_2)^2(g_2 - f_1) = 0, \]

\[ f \{f_1(f_1f_{22} + 2f_2g_{22}) + 4g_2[f_{22}(g_2 - f_1) - f_2g_{22}]\} \]

\[ - f_2^2 \left[ \frac{2((f_1)^2 + (g_2)^2) + f(f_{11} - 2g_{12}) + f_{21}g_1 - 5f_1g_2}{[f_2(g_2 - f_1) + f(f_{12} - g_{22})]^2} \right] = 0. \]

11.2 The equations \( u_{tt} - u_{xx} = f(u, u_t, u_x) \)

It shown in [38] that Eqs. (4.15),

\[ u_{tt} - u_{xx} = f(u, u_t, u_x) \]

have the following first-order invariant:

\[ \frac{2f - (u_t - u_x)(f_{u_t} - f_{u_x})}{2f - (u_t + u_x)(f_{u_t} + f_{u_x})}, \]

and two individually invariant equations:

\[ (u_t - u_x)(f_{u_t} - f_{u_x}) - 2f = 0 \]

and

\[ (u_t + u_x)(f_{u_t} + f_{u_x}) - 2f = 0. \]
12 Invariants of generalised Burgers equations

It is shown in [30], that the generalised Burgers equation (5.1):

\[ u_t + uu_x + f(t)u_{xx} = 0, \]

has for its minimal-order invariant a third-order invariant, namely the Schwarzian

\[ J = \frac{f^2}{f'^3} \left[ f''' - \frac{3}{2} \frac{f''^2}{f'} \right]. \]

Moreover, it has the following invariant differentiation:

\[ D_t = \frac{f}{f'} D_t. \]

It is restricted to differentiation with respect to \( t \) since \( f \) depends only on \( t \).

The following generalization of the Burgers equation is also considered in [30]:

\[ u_t + uu_x + g(t, x)u_{xx} = 0. \] (12.1)

It is shown that the generators (5.5) span the equivalence algebra for Eq. (12.1) as well and have the following invariants:

\[ J_1 = \frac{f_x^2}{f f_{xx}}, \quad J_2 = \frac{f^2}{f_{xx}^3} (2f_t f_x f_{tx} - f_t^2 f_{xx} - f_x^2 f_{tt}). \]
Bibliography


[9] L. Petren, *Extension de la méthode de Laplace aux équations* \[ \sum_{i=0}^{n-1} A_{1i}(x, y) \frac{\partial^{i+1} z}{\partial x^{i} \partial y^{i+1}} + \sum_{i=0}^{n} A_{0i}(x, y) \frac{\partial z}{\partial y^{i}} = 0. \] Lund: Lunds Universitets Årsskrift. N. F. Afd.2. Bd. 7. Nr 3, Kongl. Fysiografiska Sällskapets Handlingar. N.F. Bd. 22. Nr 3, 1911.
Equivalence groups and invariants of linear and non-linear equations


Linearization of third-order ordinary differential equations by point transformations

NAIL H. IBRAGIMOV
ALGA, Blekinge Institute of Technology
SE-371 79 Karlskrona, Sweden

SERGEY V. MELESHKO
School of Mathematics,
Suranaree University of Technology,
Nakhon Ratchasima, 3000, Thailand

(Received 19 May 2004; accepted 18 June 2004)

Abstract. In 1883, S. Lie found the general form of all ordinary differential equations of the second order that can be reduced to the linear equation by point transformations of the independent and dependent variables. He showed that any linearizable second-order equation should be at most cubic in the first-order derivative and provided the linearization test in terms of its coefficients.

S. Lie also noted in 1896 that third-order ordinary differential equations that can be linearized by contact transformations should be at most cubic in the second-order derivative. He himself did not investigate further the problem of linearization of third-order equations neither by contact nor by point transformations.

We present here the necessary and sufficient conditions for linearization of third-order equations by means of point transformations. We show that all third-order equations that are linearizable by point transformations are contained either in the class of equations which is linear in the second-order derivative, or in the class of equations which is quadratic in the second-order derivative. We provide the linearization test for each of these classes and describe the procedure for obtaining the linearizing point transformations as well as the linearized equation.

Keywords: Nonlinear ordinary differential equations, candidates for linearization, linearization test.
1 Introduction

1.1 Second-order equations: Lie’s linearization test

Here we outline the linearization test for second-order ordinary differential equations

\[ y'' = f(x, y, y') \]  (1.1)

due to S. Lie ([1], §1). Recall that any linear equation of the second order can be reduced to the simplest form

\[ \frac{d^2 u}{dt^2} = 0 \]  (1.2)

by a proper change of variables. Therefore any linearizable equations (1.1) is obtained from (1.2) by a change of variables

\[ t = \varphi(x, y), \quad u = \psi(x, y). \]  (1.3)

Considering \( t \) and \( u \) as the new independent and dependent variables, respectively, one obtains the following transformation of the first-order derivative:

\[ \frac{du}{dt} = \frac{D_x(\psi)}{D_x(\varphi)} = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y}, \]  (1.4)

where \( \varphi_x = \partial \varphi / \partial x, \varphi_y = \partial \varphi / \partial y \), etc., and

\[ D_x = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + y''' \frac{\partial}{\partial y''} + \cdots \]

is the total derivative. Likewise, one obtains the transformation of derivatives of the second and higher order. Namely, denoting by \( P(x, y, y') \) the right-hand side of (1.4),

\[ P(x, y, y') = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y}, \]

one has

\[ \frac{d^2 u}{dt^2} = \frac{D_x(P)}{D_x(\varphi)} = \frac{D_x(\varphi) D_x^2(\psi) - D_x(\psi) D_x^2(\varphi)}{[D_x(\varphi)]^3} \]  (1.5)

We have to substitute (1.5) in equation (1.2). We work out the expressions for the successive derivatives in (1.5):

\[ D_x(\varphi) = \varphi_x + y'\varphi_y, \quad D_x^2(\varphi) = \varphi_{xx} + 2y'\varphi_{xy} + y''\varphi_y \]
\[ D_x(\psi) = \psi_x + y'\psi_y, \quad D_x^2(\psi) = \psi_{xx} + 2y'\psi_{xy} + y''\psi_y \]  (1.6)
and collect the like terms in powers of $y'$. Then (1.5) yields the following non-linear second-order equation of the form

$$y'' + F_3(x, y)y'^3 + F_2(x, y)y'^2 + F_1(x, y)y' + F(x, y) = 0,$$

(1.7)

where we used the notation

$$F_3(x, y) = \frac{\varphi_y \psi_{yy} - \psi_y \varphi_{yy}}{\varphi_x \psi_y - \varphi_y \psi_x},$$

$$F_2(x, y) = \frac{\varphi_x \psi_{yy} - \psi_x \varphi_{yy} + 2(\varphi_y \psi_{xy} - \psi_y \varphi_{xy})}{\varphi_x \psi_y - \varphi_y \psi_x},$$

$$F_1(x, y) = \frac{\varphi_y \psi_{xx} - \psi_y \varphi_{xx} + 2(\varphi_x \psi_{xy} - \psi_x \varphi_{xy})}{\varphi_x \psi_y - \varphi_y \psi_x},$$

$$F(x, y) = \frac{\varphi_x \psi_{xx} - \psi_x \varphi_{xx}}{\varphi_x \psi_y - \varphi_y \psi_x}.$$

Thus, the linearizable second-order equations are at most cubic in the first derivative, i.e. belong to the family of equations of the form (1.7). However, not every equation of the form (1.7) with arbitrary coefficients $F_3(x, y), \ldots, F(x, y)$ is linearizable. The linearization is possible if and only if the over-determined system of non-linear partial differential equations (1.8) for two function $\varphi(x, y)$ and $\psi(x, y)$ with given $F_3(x, y), \ldots, F(x, y)$ is integrable. Lie proved\(^3\) that the equations (1.8) are integrable if and only if the following over-determined auxiliary system for $w$ and $z$ is compatible:

$$\frac{\partial w}{\partial x} = zw - FF_3 - \frac{1}{3} \frac{\partial F_1}{\partial y} + \frac{2}{3} \frac{\partial F_2}{\partial x},$$

$$\frac{\partial w}{\partial y} = -w^2 + F_2w + F_3z + \frac{\partial F_3}{\partial x} - F_1F_3,$$

$$\frac{\partial z}{\partial x} = z^2 - Fw - F_1z + \frac{\partial F}{\partial y} + FF_2,$$

$$\frac{\partial z}{\partial y} = -zw + FF_3 - \frac{1}{3} \frac{\partial F_2}{\partial x} + \frac{2}{3} \frac{\partial F_1}{\partial y}.$$  

The compatibility conditions of the auxiliary system have the form

$$3(F_3)_{xx} - 2(F_2)_{xy} + (F_1)_{yy} = (3F_1F_3 - F_2^2)_x - 3(FF_3)_y - 3F_3F_y + F_2(F_1)_y,$$

$$3F_{yy} - 2(F_1)_{xy} + (F_2)_{xx} = 3(FF_3)_x - 3(FF_2 - F_1^2)_y + 3F(F_3)_x - F_1(F_2)_x,$$

(1.9)

where the subscripts $x$ and $y$ denote differentiations in $x$ and $y$, respectively.

\(^3\)See [1], §1; reprinted in Lie’s Gessammelte Abhandlungen, vol. 5, pp. 363-370 and Note 1 on p. 423; a simple proof of Lie’s theorem is provided in [2].
Lie’s theorem furnishes us with a practical method for linearization. Namely, given an equation of the form (1.7), one should first check that its coefficients $F_3(x, y)$, $F_2(x, y)$, $F_1(x, y)$ and $F(x, y)$ obey the equations (1.9). If these conditions are satisfied, the over-determined system (1.8) is integrable and furnishes a transformation (1.3) mapping the equation (1.7) to the linear equation (1.2).

1.2 Third-order equations: Candidates for linearization

S. Lie also noted that all second-order equations can be mapped to each other by means of contact transformations ($f$, $g$, and $h$ obey the contact condition)

$$\begin{align*}
\bar{x} &= f(x, y, y'), \\
\bar{y} &= g(x, y, y'), \\
\bar{y}' &= h(x, y, y'),
\end{align*}$$

(1.10)

and that it is not true for the third-order equations. As an example, he showed (see [3], Chapter 3, §3, p. 85) that not every third-order ordinary differential equation is connected with the simplest third-order equation

$$\bar{y}'' = 0.$$  

(1.11)

Namely, the third-order ordinary differential equations related with equation (1.11) via general contact transformations (1.10) are at most cubic in the second-order derivative:

$$y'' + \Phi_3(x, y, y')y''' + \Phi_2(x, y, y')y'' + \Phi_1(x, y, y')y' + \Phi(x, y, y') = 0.$$  

(1.12)

As far as we know, Lie did not investigate further the problem of linearization of third-order equations neither by contact nor by point transformations.

In the present paper, we solve the problem of linearization of third-order equations

$$y''' = f(x, y, y', y'')$$

(1.13)

by means of point transformations (1.2). First, we have to find all candidates for linearization, i.e. the general form of third-order equations (1.13) that can be obtained from linear equations by any point transformation (1.2).

E. Laguerre showed in 1879 that in any linear ordinary differential equation of order $n \geq 3$ the two terms of orders next below the highest can be simultaneously removed by an equivalence transformation (see [4], Section 10.2.1 and the references therein). Therefore, without loss of generality, any linear third-order equation with the independent variable $t$ and the dependent variable $u$ can be written in the form

$$u'' + \alpha(t)u = 0.$$  

(1.14)

Denoting by $Q(x, y, y', y'')$ the right-hand side of (1.5), using (1.6) and singling out the terms with $y''$, we have

$$Q = \frac{\varphi_x y' + \varphi_y y''}{(\varphi_x + y' \varphi_y)} y'' + \cdots.$$  

(1.15)
Hence,
\[ u''' = \frac{D_x(Q)}{D_x(\varphi)} = \frac{\Delta}{(\varphi_x + y'\varphi_y)^3} \left[ - (\varphi_x + y'\varphi_y) y''' + 3\varphi_y (y'')^2 \right] + \cdots \] (1.16)
the other terms being at most linear in \( y'' \). Here \( \varphi_x = \partial \varphi / \partial y, \varphi_y = \partial \varphi / \partial y, \ldots, \) and
\[ \Delta = \varphi_y \psi_x - \varphi_x \psi_y \neq 0 \] (1.17)
is the Jacobian of the change of variables (1.3). It is manifest from Equation (1.16) that the transformations (1.3) with \( \varphi_y = 0 \) and \( \varphi_y \neq 0 \), respectively, provide two distinctly different candidates for linearization (cf. Equation (1.12)).

If \( \varphi_y = 0 \) we work out the missing terms in (1.16), substitute the resulting expression in (1.14) and obtain the following first candidate for linearization:
\[ y''' + (A_1 y' + A_0)y'' + B_3 y'^3 + B_2 y'^2 + B_1 y' + B_0 = 0, \] (1.18)
where \( A_i = A_i(x, y) \) and \( B_i = B_i(x, y) \) are arbitrary functions of \( x, y \).

If \( \varphi_y \neq 0 \), we proceed likewise and, setting \( r(x, y) = \varphi_x / \varphi_y \), arrive at the second candidate for linearization:
\[ y''' + \frac{1}{y'} + r \left[ -3(y'')^2 + (C_2 y'^2 + C_1 y' + C_0) y'' \right. \]
\[ \left. + D_5 y'^5 + D_4 y'^4 + D_3 y'^3 + D_2 y'^2 + D_1 y' + D_0 \right] = 0, \] (1.19)
where \( r = r(x, y), C_i = C_i(x, y) \) and \( D_i = D_i(x, y) \) are arbitrary functions of \( x, y \).

The problem of linearization by point transformations for Equation (1.18) was discussed in [5], [6] but was not completely solved (see further Remark 2.3). To the best of our knowledge, the problem for the second candidate, Equation (1.19), has not been considered yet.

2 Formulation of the linearization theorems

We have shown in the previous section that every linearizable third-order equation belongs either to the class of equations (1.19) that are at most quadratic in \( y'' \) with a specific dependence on \( y' \), or to the class of equations (1.18) with the linear dependence on the second-order derivative.

In this section, we formulate the main theorems containing necessary and sufficient conditions for linearization as well as the methods for constructing the linearizing transformations for each candidate. Proofs of the theorems and illustrative examples are provided in the subsequent sections.
2.1 The linearization test for Equation (1.18)

Recall that the linearization test for the second-order equation (1.7) is provided by two differential equations (1.9) for four coefficients of equation (1.7). The linearizing change of variables is obtained by solving four differential equations (1.8) with respect to two functions \( \varphi(x, y) \) and \( \psi(x, y) \).

Consider the first candidate for linearization, i.e. Equation (1.18). In this case, the linearizing transformations (1.3) have the form

\[
\begin{align*}
  t &= \varphi(x), \\
  u &= \psi(x, y).
\end{align*}
\]

Theorem 2.1. Equation (1.18)

\[
y''' + (A_1 y' + A_0) y'' + B_3 y' + B_2 y^2 + B_1 y' + B_0 = 0
\]

is linearizable if and only if its coefficients obey the following five equations:

\[
\begin{align*}
  &A_0 y - A_{1x} = 0, \quad (3B_1 - A_0^2 - 3A_{0x}) y = 0, \quad (2.3) \\
  &3B_2 = 3A_{1x} + A_0 A_1, \quad 9B_3 = 3A_{1y} + A_1^2, \quad (2.4) \\
  &27B_{0yy} = (9B_1 - 6A_{0x} - 2A_0^2) A_{1x} + 9(B_{1x} - A_1 B_0) y + 3B_{1y} A_0. \quad (2.5)
\end{align*}
\]

Provided that the conditions (2.3)-(2.5) are satisfied, the linearizing transformation (2.1) is defined by a third-order ordinary differential equation for the function \( \varphi(x) \), namely by the Riccati equation

\[
6 \frac{d\chi}{dx} - 3\chi^2 = 3B_1 - A_0^2 - 3A_{0x}
\]

for

\[
\chi = \frac{\varphi_{xx}}{\varphi_x},
\]

and by the following integrable system of partial differential equations for \( \psi(x, y) \):

\[
\begin{align*}
  3\psi_{yy} &= A_1 \psi_y, \\
  3\psi_{xy} &= (3\chi + A_0) \psi_y, \\
  \psi_{xxx} &= 3\chi \psi_{xx} + B_0 \psi_y - \frac{1}{6} (3A_{0xx} + A_0^2 - 3B_1 + 9\chi^2) \psi_x - \Omega \psi,
\end{align*}
\]

where \( \chi \) is given by (2.7) and \( \Omega \) is the following expression:

\[
\Omega = \frac{1}{54} \left( 9A_{0xx} + 18A_{0x} A_0 + 54B_{0y} - 27B_{1x} + 4A_0^3 - 18A_0 B_1 + 18A_1 B_0 \right). \quad (2.10)
\]

Finally, the coefficient \( \alpha \) of the resulting linear equation (1.14) is given by

\[
\alpha = \Omega \varphi_x^{-3}. \quad (2.11)
\]
Remark 2.1. Using the equations (2.3) and (2.4), one can replace Equation (2.5) by the equation
\[ \Omega_y = 0. \]  
(2.12)
The latter equation follows from Equation (2.11) since \( \varphi_y = 0 \) and hence \( (\alpha(t))_y = (\alpha(\varphi(x)))_y = 0 \).

Remark 2.2. Let us assume that \( \Omega \neq 0 \). Then one can find the expressions for \( \varphi_{xx} \) and \( \varphi_{xxx} \) from Equation (2.11) and substitute them into Equation (2.6) to obtain
\[ \alpha^{-8/3} \Omega^{8/3} (6\alpha' - 7(\alpha')^2) = 6\Omega_{xx} \Omega - 7\Omega_x^2 - 9\beta^2 \]  
(2.13)where \( \beta = (3B_1 - A_0^2 - 3A_0)/3 \) and \( \alpha' \) is the derivative of the function \( \alpha(t) \) with respect to \( t \).

Remark 2.3. In case \( \Omega = 0 \), the necessary and sufficient conditions for linearization given by our four equations (2.3), (2.4) together with the fifth equation \( \Omega = 0 \) are equivalent to the five equations given by (22), (23), (24) and (21) from [6].

In case \( \Omega \neq 0 \), the conditions (22), (23), (26) and (21) are given in [6] as the necessary and sufficient conditions for linearization. However, upon examining simple examples (e.g. the equation \( y'' + y^2 = 0 \); see Example 5.3) this statement appears false. To complete the linearization test, our Equation (2.5) or the equivalent equation (2.12) should be added. Note that our Equation (2.13) is equivalent to the equation (26) from [6] after substituting the factor 7 which is missing in [6].

2.2 The linearization test for Equation (1.19)
The following theorem provides the test for linearization of the second candidate. The necessary and sufficient conditions comprise eight differential equations (2.14)-(2.21) for ten coefficients of the equation (1.19). The linearizing change of variables (1.3) is determined by Equations (2.23)-(2.26) for the functions \( \varphi(x, y) \) and \( \psi(x, y) \).

Theorem 2.2. Equation (1.19) is linearizable if and only if its coefficients obey the following equations:
\[ C_0 = 6r \frac{\partial r}{\partial y} - 6 \frac{\partial r}{\partial x} + r C_1 - r^2 C_2, \]  
(2.14)
\[ 6 \frac{\partial^2 r}{\partial y^2} = \frac{\partial C_2}{\partial x} - \frac{\partial C_1}{\partial y} + r \frac{\partial C_2}{\partial y} + C_2 \frac{\partial r}{\partial y}, \]  
(2.15)
and

\[
18D_0 = 3r^3 \frac{\partial C_1}{\partial y} - 6r^2 \frac{\partial C_1}{\partial x} - 3r^3 \frac{\partial C_2}{\partial x} + 9r^4 \frac{\partial C_2}{\partial y} - 36r^2 \frac{\partial^2 r}{\partial x \partial y}
\]

\[
+ 18r \frac{\partial^2 r}{\partial x^2} - 54 \left( \frac{\partial r}{\partial x} \right)^2 + 90r \frac{\partial r}{\partial x} \frac{\partial r}{\partial y} - 36r^2 \left( \frac{\partial r}{\partial y} \right)^2 + 6r(3C_1 - rC_2) \frac{\partial r}{\partial x}
\]

\[
+ 9r^2 (rC_2 - 2C_1) \frac{\partial r}{\partial y} - 2r^2 C_1^2 + 2r^3 C_1 C_2 + 4r^4 C_2^2 + 18r^4 D_4 - 72r^5 D_5,
\]

(2.16)

\[
18D_1 = 9r^2 \frac{\partial C_1}{\partial y} - 12r \frac{\partial C_1}{\partial x} - 27r^2 \frac{\partial C_2}{\partial x} + 33r^3 \frac{\partial C_2}{\partial y} - 36r \frac{\partial^2 r}{\partial x \partial y}
\]

\[
+ 18 \frac{\partial^2 r}{\partial x^2} + 6(3C_1 + 4rC_2) \frac{\partial r}{\partial x} - 3r(6C_1 + 7rC_2) \frac{\partial r}{\partial y} + 18r \left( \frac{\partial r}{\partial y} \right)^2
\]

\[
- 18 \frac{\partial r}{\partial x} \frac{\partial r}{\partial y} - 4rC_1^2 - 2r^2 C_1 C_2 + 20r^3 C_2^2 + 72r^3 D_4 - 270r^4 D_5,
\]

(2.17)

\[
9D_2 = 3r \frac{\partial C_1}{\partial y} - 3 \frac{\partial C_1}{\partial x} - 21r \frac{\partial C_2}{\partial x} + 21r^2 \frac{\partial C_2}{\partial y} + 15C_2 \frac{\partial r}{\partial x}
\]

\[
- 15r C_2 \frac{\partial r}{\partial y} - C_1^2 - 5r C_1 C_2 + 14r^2 C_2^2 + 54r^2 D_4 - 180r^3 D_5,
\]

(2.18)

\[
3D_3 = 3r \frac{\partial C_2}{\partial y} - 3 \frac{\partial C_2}{\partial x} - C_1 C_2 + 2r C_2^2 + 12r D_4 - 30r^2 D_5,
\]

(2.19)

\[
54 \frac{\partial D_4}{\partial x} = 18 \frac{\partial^2 C_1}{\partial y^2} + 3C_2 \frac{\partial C_1}{\partial y} - 72 \frac{\partial^2 C_2}{\partial x \partial y} - 39C_2 \frac{\partial C_2}{\partial x}
\]

\[
+ 18r \frac{\partial^2 C_2}{\partial y^2} - 3rC_2 \frac{\partial C_2}{\partial y} + \left( 72 \frac{\partial C_2}{\partial y} + 33C_2^2 \right) \frac{\partial r}{\partial y} + 108 D_4 \frac{\partial r}{\partial y}
\]

\[
+ 270 D_5 \frac{\partial r}{\partial x} + 378 r \frac{\partial D_5}{\partial x} - 108r^2 \frac{\partial D_5}{\partial y} - 540 r D_5 \frac{\partial r}{\partial y}
\]

\[
+ 36 r C_1 D_5 - 8 r C_3 + 36 r C_2 D_4 + 108 r^2 C_2 D_5 + 54 r H,
\]

(2.20)

and

\[
\frac{\partial H}{\partial x} = 3H \frac{\partial r}{\partial y} + r \frac{\partial H}{\partial y},
\]

(2.21)

where

\[
H = \frac{\partial D_4}{\partial y} - 2 \frac{\partial D_5}{\partial x} - 3r \frac{\partial D_5}{\partial y} - 5D_5 \frac{\partial r}{\partial y} - 2r C_2 D_5
\]

\[
+ \frac{1}{3} \left[ \frac{\partial^2 C_2}{\partial y^2} + 2C_2 \frac{\partial C_2}{\partial y} - 2C_1 D_5 + 2C_2 D_4 \right] + \frac{4}{27} C_2^3
\]

(2.22)
Linearization of third-order equations

Provided that the conditions (2.14)-(2.21) are satisfied, the transformation (1.3) mapping equation (1.19) to a linear equation (1.14) is obtained by solving the following compatible system of equations for the functions \( \varphi(x, y) \) and \( \psi(x, y) \):

\[
\frac{\partial \varphi}{\partial x} = r \frac{\partial \varphi}{\partial y}, \quad \frac{\partial \psi}{\partial x} = -\frac{\partial \varphi}{\partial y} W + r \frac{\partial \psi}{\partial y}, \quad (2.23)
\]

\[
6 \frac{\partial \varphi}{\partial y} \frac{\partial^3 \varphi}{\partial y^3} = 9 \left( \frac{\partial^2 \varphi}{\partial y^2} \right)^2 + \left[ 15 r D_5 - 3 D_4 - C_2^2 - 3 \frac{\partial C_2}{\partial y} \right] \left( \frac{\partial \varphi}{\partial y} \right)^2, \quad (2.24)
\]

\[
\frac{\partial^3 \psi}{\partial y^3} = W \frac{\partial \varphi}{\partial y} + \frac{1}{6} \left[ 15 r D_5 - C_2^2 - 3 D_4 - 3 \frac{\partial C_2}{\partial y} \right] \frac{\partial \psi}{\partial y} - \frac{1}{2} H \psi + 3 \frac{\partial^2 \varphi}{\partial y^2} \frac{\partial^2 \psi}{\partial y^2} \left( \frac{\partial \varphi}{\partial y} \right)^{-1} - \frac{3}{2} \left( \frac{\partial^2 \varphi}{\partial y^2} \right)^2 \frac{\partial \psi}{\partial y} \frac{\partial \varphi}{\partial y}^{-2}, \quad (2.25)
\]

where the function \( W \) is defined by the equations

\[
3 \frac{\partial W}{\partial x} = \left[ C_1 - r C_2 + 6 \frac{\partial r}{\partial y} \right] W, \quad 3 \frac{\partial W}{\partial y} = C_2 W. \quad (2.26)
\]

The coefficient \( \alpha \) of the resulting linear equation (1.14) is given by (cf. (2.11))

\[
\alpha = \frac{H}{2(\varphi_y)^3}, \quad (2.27)
\]

where \( H \) is the function defined in (2.22).

3 Relations between coefficients and transformations

Recall that the linearization theorem for second-order equations is based on the relations (1.8) connecting the coefficients of equation (1.7) with the functions \( \varphi(x, y) \) and \( \psi(x, y) \) of the transformation (1.3). Equations (1.8) are obtained by using the transformation formulae (1.5) and (1.6). Now we need the similar relations between \( \varphi(x, y), \psi(x, y) \) and the coefficients of two candidates for linearization, (1.18) and (1.19), respectively. To obtain these relations, we complete the equations (1.15)-(1.16) by adding the missing terms, substitute the result in the linear equation (1.14).
3.1 The coefficients of Equation (1.18)

Lemma 3.1. The coefficients of equation (1.18) and the functions $\varphi(x)$ and $\psi(x, y)$ in the transformation (2.1) are related by the following equations:

$$
A_1 = 3 \psi_y^{-1} \psi_{yy}, \quad A_0 = 3 (\varphi_x \psi_y)^{-1}(\varphi_x \psi_{xy} - \psi_y \varphi_{xx}), \quad (3.1)
$$

$$
B_3 = \psi_y^{-1} \psi_{yyy}, \quad B_2 = 3 (\varphi_x \psi_y)^{-1}(\varphi_x \psi_{xyy} - \psi_{yy} \varphi_{xx}), \quad (3.2)
$$

$$
B_1 = (\varphi_x^2 \psi_y)^{-1}(3 \varphi_{xx}^2 \psi_y - \varphi_{xxx} \varphi_x \psi_y - 6 \varphi_{xx} \varphi_x \psi_{xy} + 3 \varphi_x^2 \psi_{xyy}), \quad (3.3)
$$

$$
B_0 = (\varphi_x^2 \psi_y)^{-1}(3 \varphi_{xx}^2 \psi_x - \varphi_{xxx} \varphi_x \psi_x - 3 \varphi_{xx} \varphi_x \psi_{xx} + \varphi_x^2 \psi_{xxx} + \alpha \varphi_x^5). \quad (3.4)
$$

3.2 The coefficients of Equation (1.19)

Lemma 3.2. The coefficients of the equation (1.19) and the functions $\varphi(x, y)$ and $\psi(x, y)$ in the transformation (1.3) are related by the following equations:

$$
C_2 = 3 \frac{\partial \varphi}{\partial y} \Delta^{-1} \left\{ \frac{\partial \varphi}{\partial y} \frac{\partial \Delta}{\partial y} - 2 \frac{\partial^2 \varphi}{\partial y^2} \Delta \right\}, \quad (3.5)
$$

$$
C_1 = 3 \frac{\partial \varphi}{\partial y} \Delta^{-1} \left\{ \left( \frac{r \partial \Delta}{\partial y} + \frac{\partial \Delta}{\partial x} \right) \frac{\partial \varphi}{\partial y} - 4 \Delta \frac{\partial}{\partial y} \left( \frac{r \partial \varphi}{\partial y} \right) \right\}, \quad (3.6)
$$

$$
C_0 = 3 \frac{\partial \varphi}{\partial y} \Delta^{-1} \left\{ \frac{r \partial \varphi}{\partial y} \frac{\partial \Delta}{\partial x} - 2 \frac{\partial^2 \varphi}{\partial y^2} r^2 \Delta - 2 \frac{\partial \varphi}{\partial y} \frac{\partial r}{\partial x} \Delta - 2 r \frac{\partial \varphi}{\partial y} \frac{\partial r}{\partial y} \Delta \right\}, \quad (3.7)
$$

$$
D_5 = \left[ \frac{\partial \varphi}{\partial y} \Delta \right]^{-1} \left\{ \frac{\partial \varphi}{\partial y} \left( \frac{\partial^3 \varphi}{\partial y^3} - \frac{\partial^3 \varphi}{\partial y^2 \partial y^2} \right) + \alpha \varphi \left( \frac{\partial \varphi}{\partial y} \right)^5 \right\}, \quad (3.8)
$$

$$
D_4 = \left[ \frac{\partial \varphi}{\partial y} \Delta \right]^{-1} \left\{ \left[ 4 \Delta - 5r \frac{\partial \varphi}{\partial y} \frac{\partial \psi}{\partial y} \right] \frac{\partial^3 \varphi}{\partial y^3} \right\} + 3 \left[ 5r \frac{\partial^2 \varphi}{\partial y^2} \frac{\partial \psi}{\partial y} - 5 \frac{\partial^2 \varphi}{\partial y^2} \frac{\partial \psi}{\partial y} \right]^{-1} - 5r \frac{\partial \varphi}{\partial y} \frac{\partial^2 \psi}{\partial y^2} + 4 \frac{\partial \Delta}{\partial y} \frac{\partial^2 \varphi}{\partial y^2} \right\} \quad (3.9)
$$
\[ D_3 = \left[ \frac{\partial \varphi}{\partial y} \Delta \right]^{-1} \left\{ 2r \left[ 8\Delta - 5r \frac{\partial \varphi}{\partial y} \frac{\partial \psi}{\partial y} \frac{\partial \varphi}{\partial y^3} \right] + 6 \left( \frac{\partial \Delta}{\partial x} + 7r \frac{\partial \Delta}{\partial y} - 3\Delta \frac{\partial r}{\partial y} \right) \right. \\
+ 30r \left( r \frac{\partial^2 \varphi}{\partial y^2} \frac{\partial \psi}{\partial y} - 2\Delta \frac{\partial^2 \varphi}{\partial y^2} \left( \frac{\partial \varphi}{\partial y} \right)^{-1} - r \frac{\partial \varphi}{\partial y} \frac{\partial^2 \psi}{\partial y^2} \right) \left. \right\} \frac{\partial^2 \varphi}{\partial y^2} \]

(3.10)

\[ D_2 = \left[ \frac{\partial \varphi}{\partial y} \Delta \right]^{-1} \left\{ 2r^2 \left[ 12\Delta - 5r \frac{\partial \varphi}{\partial y} \frac{\partial \psi}{\partial y} \frac{\partial^3 \varphi}{\partial y^3} \right] + 10r^3 \left( \frac{\partial \varphi}{\partial y} \right)^2 \frac{\partial^3 \psi}{\partial y^3} \right. \\
+ 6 \left[ 3r \frac{\partial \Delta}{\partial x} + 9r^2 \frac{\partial \Delta}{\partial y} - 5r \frac{\partial \varphi}{\partial y} \frac{\partial^2 \varphi}{\partial y^2} \right] - \Delta \frac{\partial r}{\partial x} - 8r \Delta \frac{\partial r}{\partial y} \frac{\partial \varphi}{\partial y} \right. \\
+ 30r^2 \left[ r \frac{\partial \psi}{\partial y} - 3\Delta \left( \frac{\partial \varphi}{\partial y} \right)^{-1} \right] \left( \frac{\partial^2 \varphi}{\partial y^2} \right)^2 + 10\alpha \psi r^3 \left( \frac{\partial \varphi}{\partial y} \right)^5 \left. \right\} \frac{\partial \varphi}{\partial y} \]

(3.11)

\[ D_1 = \left[ \frac{\partial \varphi}{\partial y} \Delta \right]^{-1} \left\{ r^3 \left[ 16\Delta - 5r \frac{\partial \varphi}{\partial y} \frac{\partial \psi}{\partial y} \right] \frac{\partial^3 \varphi}{\partial y^3} + 5r^4 \left( \frac{\partial \varphi}{\partial y} \right)^2 \frac{\partial^3 \psi}{\partial y^3} + \alpha \psi \left( \frac{\partial \varphi}{\partial y} \right)^3 \right. \\
+ 3r \left[ 6r \frac{\partial \Delta}{\partial x} + 10r^2 \frac{\partial \Delta}{\partial y} - 5r^3 \frac{\partial \varphi}{\partial y} \frac{\partial^2 \varphi}{\partial y^2} - 4\Delta \frac{\partial r}{\partial x} - 14r \Delta \frac{\partial \varphi}{\partial y} \frac{\partial \varphi}{\partial y} \right. \\
+ 15r^3 \left( r \frac{\partial \psi}{\partial y} - 4\Delta \left( \frac{\partial \varphi}{\partial y} \right)^{-1} \right) \left( \frac{\partial^2 \varphi}{\partial y^2} \right)^2 + \left[ \left( \frac{\partial^2 r}{\partial x^2} + 6r \frac{\partial^2 \varphi}{\partial x \partial y} + 11r^2 \frac{\partial^3 \varphi}{\partial x \partial y^2} \right) \frac{\partial \varphi}{\partial y} \right. \left( \frac{\partial \varphi}{\partial y} \right)^2 \]

(3.12)
4 Proof of the linearization theorems

The proof of the linearization theorems formulated in Section 2 requires investigation of integrability conditions for the equations given in Section 3. We will consider the problem for the candidates (1.18) and (1.19) for linearization separately. The problem is formulated as follows. Given the coefficients $A_i(x, y)$, $B_i(x, y)$, and $C_i(x, y)$, $D_i(x, y)$ of the equations (1.18) and (1.19), respectively, find the integrability conditions of the respective equations for the functions $\varphi(x)$ and $\psi$.

4.1 Equation (1.18)

Let us prove Theorem 2.1 on the linearization of Equation (1.18). Namely, given the coefficients $A_i(x, y)$, $B_i(x, y)$ of Equation (1.18), we have to find the necessary and sufficient conditions for integrability of the over-determined system (3.1)-(3.4) for the unknown functions $\varphi(x)$ and $\psi(x, y)$.

We first rewrite the expressions (3.1) for $A_0$ and $A_1$ as follows:

$$A_0 = 3 \frac{W_x}{W}, \quad A_1 = 3 \frac{H_y}{H},$$

where $W = \psi_y/\varphi_x$ and

$$H = \psi_y.$$  \hspace{1cm} (4.1)

Equation (4.1) and the definition of $W$ yield:

$$\varphi_x = HW^{-1}.$$  \hspace{1cm} (4.2)
Differentiation of Equation (4.2) with respect to $y$ yields:

$$W_y = H_y W H^{-1}. \quad (4.3)$$

Now the equations (3.2) and (3.3) are written in the form

$$B_3 = H^{-1} H_{yy}, \quad B_2 = 3W^{-1}H^{-2}(H_{xy} HW - H_x H_y W + H_y W_x H)$$

and

$$B_1 = (HW)^{-2}(2H_x H/W^2 - 3H_x^2 W^2 + 2H_x W_x HW + W_{xx} H^2 W + W_x^2 H^2),$$

respectively. Furthermore, Equation (3.4) for $B_0$ becomes

$$S \equiv (H^2 W_{xx} - HW^2 H_{xx} + 3W^2 H_x^2 - 4HW H_x W_x + H^2 W_x^2)W\psi_x + 3(H W_x - W H_x) W^2 \psi_{xx} + H^2 W^3 \psi_{xxx} - H^3 W^3 B_0 + \alpha H^5 \psi = 0. \quad (4.4)$$

One can determine $\alpha$ from Equation (4.4). Namely, the reckoning shows that it is convenient to use, instead of $S = 0$, the equation

$$H S_y - 5H_y S = 0.$$

It follows:

$$\alpha = \frac{W}{H^6} \left( H^3 W^2 B_{0y} - H_{xxx} H^2 W^2 + 4H_x H_x HW^2 - 3H_{xx} W_x H^2 W - 3H_x^2 W^2 + 4H_x W_x HW - H_x W_{xx} H^2 W - H_x W_x^2 H^2 + H_y B_0 H^2 W^2 \right). \quad (4.5)$$

Since $\varphi = \varphi(x)$, we have $\alpha_y = 0$ and Equation (4.5) yields:

$$H^3 B_{0yy} + H^2 H_y B_{0y} + H^3 (H^{-1} H)_y B_0 - [3H_{xy} H_x^2 - H H_{xy} H_{xx} + H^2 H_{xxx} H_y - 3H_{xy} H_x H_y + 4H_x H_x H_y - 3H^{-1} H_x^3 H_y] - W^{-1}[4H_x^2 H_y W_x - 4H H_x H_y W_{xx} - 4H H_{xx} H_{xy} H_y W_x - 3H_{xx} H_{xx} H_y W_x + 3H^2 H_{xx} H_{xy} W_x] - HW^{-2} W_x[H H_{xy} - H_y H_x] = 0. \quad (4.6)$$

Rewriting Equation (4.2) in the form

$$H = W \varphi_x \quad (4.7)$$

and invoking that $\varphi = \varphi(x)$, the representations for $B_2$ and $B_3$ can be written as

$$B_2 = 3W^{-1} W_{xy}, \quad B_3 = W^{-1} W_{yy}. $$
The representation for $B_1$, upon denoting $\chi = \varphi^{-1}_x \varphi_{xx}$, leads to Equation (2.6):

$$3(2\chi' - \chi^2) = 3B_1 - 3\frac{\partial A_0}{\partial x} - A_0^2.$$  \hspace{1cm} (4.8)

Using Equation (4.3) and the expressions for $A_0$ and $A_1$, one determines the first-order derivatives of $W$:

$$W_x = \frac{1}{3} W A_0, \quad W_y = \frac{1}{3} W A_1.$$ \hspace{1cm} (4.9)

Hence, the equations (2.4):

$$3B_2 = 3\frac{\partial A_1}{\partial x} + A_0 A_1, \quad 9B_3 = 3\frac{\partial A_1}{\partial y} + A_1^2.$$

Equating the mixed derivatives $W_{xy}$ and $W_{yx}$ obtained from Equations (4.9), one arrives at the first equation (2.3):

$$\frac{\partial A_0}{\partial y} = \frac{\partial A_1}{\partial x}.$$ \hspace{1cm} (4.10)

Since $\varphi$, and hence $\chi$ does not depend on $y$, differentiation of Equation (4.8) with respect to $y$ yields the second equation (2.3). Furthermore, invoking the equations (4.1), (4.9), we eliminate $H$ and $W$, together with their derivatives, from Equation (4.6) and arrive at Equation (2.5).

The equations (2.8) are provided by (4.9) whereas Equation (2.9) is obtained from Equation (4.4). Thus, we can obtain all third-order derivatives of $\psi$. Namely Equation (2.9) gives $\psi_{xxx}$, and the remaining derivatives $\psi_{xxy}, \psi_{xyy}$ and $\psi_{yyy}$ are obtained from the equations (2.8) by differentiating. The reckoning shows that all mixed fourth-order derivatives found from these different expressions for the third-order derivatives are equal. It means that the equations (2.8)-(2.9) for $\psi(x, y)$ are in involution. Finally, we obtain the equations (2.10)-(2.11) from (4.5) and complete the proof of Theorem 2.1.

### 4.2 Equation (1.19)

The problem is formulated as follows. Given the coefficients $C_i(x, y)$, $D_i(x, y)$ of the equation (1.19), find the necessary and sufficient conditions for integrability of the over-determined system of equations (3.5)-(3.13) for the unknown functions $\varphi(x, y)$ and $\psi(x, y)$. Recall that, according to our notation, the following equations hold:

$$\varphi_x = r\varphi_y, \quad \psi_x = \frac{\psi_y \varphi_x - \Delta}{\varphi_y}.$$ \hspace{1cm} (4.11)

and

$$\alpha_x = \frac{\varphi_x}{\varphi_y} \alpha_y.$$ \hspace{1cm} (4.12)
Let us simplify the expressions (3.5)-(3.7) for the coefficients $C_i$. We rewrite the right-hand side of the equation (3.5) as follows:

$$
\frac{3}{\Delta \varphi_y} \left( \Delta_y \varphi_y - 2 \Delta \varphi_{yy} \right) = \frac{3}{\Delta \varphi_y} \varphi_y^3 \frac{\partial}{\partial y} \left( \frac{\Delta}{\varphi_y^2} \right) = \frac{3 \varphi_y^2}{\Delta} \frac{\partial}{\partial y} \left( \frac{\Delta}{\varphi_y^2} \right).
$$

Therefore, we set

$$
C_2 = 3 \frac{W_y}{W} \quad (4.13)
$$

and rewrite the equation (3.5) in the form

$$
\frac{W_y}{W} = \frac{\varphi_y^2}{\Delta} \frac{\partial}{\partial y} \left( \frac{\Delta}{\varphi_y^2} \right).
$$

The integration yields

$$
W(x, y) = h(x) \frac{\Delta}{\varphi_y^2}.
$$

Since the coefficient $h(x)$ will not appear in the expression (4.13) for $C_2$, we will let $h = 1$ and get

$$
\Delta = W \varphi_y^2. \quad (4.14)
$$

Then the equations (4.13), (3.6) and (3.7) yield:

$$
C_2 = 3 \frac{W_y}{W}, \quad C_1 = 3 \frac{W_x + rW_y}{W} - 6ry, \quad C_0 = 3 \frac{rW_x - 2Wr_x}{W}. \quad (4.15)
$$

Substituting the expression (4.14) for $\Delta$ in the equations (3.8) and (3.9), one arrives at the following equations:

$$
\varphi_y^2 \varphi_{yyy} = \left( \varphi_{yy} \varphi_{yyy} \psi_y - 3 \varphi_{yy}^2 \psi_y + 3 \varphi_y \varphi_{yy} \psi_{yy} - \alpha \varphi_y^5 \psi + W \varphi_y^3 D_5 \right), \quad (4.16)
$$

$$
2W \varphi_y \varphi_{yyy} = \left( 3W \varphi_y^2 - 3 \varphi_y^2 W_y - W \varphi_y^2 D_4 + 5 \varphi_y^2 rWD_5 \right). \quad (4.17)
$$
Furthermore, the expressions (3.10)-(3.13) for \( D_3, \ldots, D_0 \) become:

\[
D_3 = W^{-1} \left[ 6 \frac{\partial r}{\partial y} \frac{\partial W}{\partial y} - 3 \frac{\partial^2 W}{\partial x \partial y} + 3 \frac{\partial^2 W}{\partial y^2} r + 4 D_4 r W - 10 D_5 r^2 W \right], \tag{4.18}
\]

\[
D_2 = W^{-1} \left[ 2 \frac{\partial^2 r}{\partial x \partial y} W + 4 \frac{\partial r}{\partial x} \frac{\partial W}{\partial y} - 2 \frac{\partial^2 r}{\partial y^2} r W - 4 \left( \frac{\partial r}{\partial y} \right)^2 W + 4 \frac{\partial r}{\partial y} \frac{\partial W}{\partial x} \right. \tag{4.19}
\]

\[
+ 10 \frac{\partial r}{\partial y} \frac{\partial W}{\partial y} - 7 \frac{\partial^2 W}{\partial x \partial y} r - \frac{\partial^2 W}{\partial x^2} r^2 + 8 \frac{\partial^2 W}{\partial y^2} r^2 + 6 D_4 r^2 W - 20 D_5 r^3 W \right], \]

\[
D_1 = W^{-1} \left[ 2 \frac{\partial^2 r}{\partial x^2} r W + \frac{\partial^2 r}{\partial x^2} W - 7 \frac{\partial r}{\partial x} \frac{\partial W}{\partial y} W + 3 \frac{\partial r}{\partial x} \frac{\partial W}{\partial x} + \frac{\partial r}{\partial y} \frac{\partial W}{\partial x} \right. \tag{4.20}
\]

\[
+ 5 \frac{\partial r}{\partial x} \frac{\partial W}{\partial y} r - 3 \frac{\partial^2 W}{\partial y^2} r^2 W - \left( \frac{\partial r}{\partial y} \right)^2 W + 5 \frac{\partial r}{\partial y} \frac{\partial W}{\partial x} r + 5 \frac{\partial r}{\partial y} \frac{\partial W}{\partial y} r^2 \]

\[
- 5 \frac{\partial^2 W}{\partial x \partial y} r^2 - 2 \frac{\partial^2 W}{\partial x^2} r + 7 \frac{\partial^2 W}{\partial y^2} r^3 + 4 D_4 r^3 W - 15 D_5 r^4 W \right], \]

\[
D_0 = W^{-1} \left[ \frac{\partial^2 r}{\partial x^2} r W - 3 \left( \frac{\partial r}{\partial x} \right)^2 W - \frac{\partial r}{\partial x} \frac{\partial W}{\partial y} W + 3 \frac{\partial r}{\partial x} \frac{\partial W}{\partial x} r \right. \tag{4.21}
\]

\[
+ \frac{\partial r}{\partial x} \frac{\partial W}{\partial y} r^2 - \frac{\partial^2 r}{\partial y^2} r^3 W + \frac{\partial r}{\partial y} \frac{\partial W}{\partial x} r^2 + \frac{\partial r}{\partial y} \frac{\partial W}{\partial y} r^3 \]

\[
- \frac{\partial^2 W}{\partial x \partial y} r^3 - \frac{\partial^2 W}{\partial x^2} r^2 + 2 \frac{\partial^2 W}{\partial y^2} r^4 + D_4 r^4 W - 4 D_5 r^5 W \right]. \]

Let us turn now to the integrability problem. One can find all third-order derivatives of the functions \( \varphi \) and \( \psi \) by using the equations (4.11), (4.16) and (4.17). Then, calculating the cross derivatives, one obtains from the equation \( (\varphi_{xyy})_y = (\varphi_{yyy})_x \) :

\[
\frac{\partial D_4}{\partial x} = \frac{\partial D_4}{\partial y} + 5 r \frac{\partial D_5}{\partial x} - 5 r^2 \frac{\partial D_5}{\partial y} + 5 D_5 \left( \frac{\partial r}{\partial x} - 3 r \frac{\partial r}{\partial y} \right) - 2 \frac{\partial^3 r}{\partial y^3} + 2 D_4 \frac{\partial r}{\partial y} \tag{4.22}
\]

\[
+ 3 W^{-1} \left[ 2 \frac{\partial r}{\partial y} \frac{\partial^2 W}{\partial x \partial y^2} - \frac{\partial^3 W}{\partial x \partial y^3} + r \frac{\partial^3 W}{\partial y^3} \right] + 3 W^{-2} \frac{\partial^2 W}{\partial y^2} \left[ \frac{\partial W}{\partial x} - r \frac{\partial W}{\partial y} \right]
\]

Furthermore, we consider the equation \( (\psi_{xyy})_y = (\psi_{yyy})_x \) and write it the form

\[
S \equiv 2 \alpha \varphi^3 - H = 0, \tag{4.23}
\]
where (cf. (2.22))

\[ H = \frac{\partial D_4}{\partial y} - 2\frac{\partial D_5}{\partial x} - 3r\frac{\partial D_5}{\partial y} - D_5\frac{\partial r}{\partial y} \]

\[ + W^{-1}\left[ \frac{\partial^3 W}{\partial y^3} - 2D_5\frac{\partial W}{\partial x} + \left(2D_4 - 8rD_5 + 3W^{-1}\frac{\partial^2 W}{\partial y^2}\right)\frac{\partial W}{\partial y}\right]. \]

Since \( \varphi_y \neq 0 \), Eq. (4.23) yields (2.27):

\[ \alpha = \frac{H}{2\varphi_y^3} \]

Now the equation \( \alpha_x - r\alpha_y = 0 \) leads to Eq. (2.21):

\[ \frac{\partial H}{\partial x} - r\frac{\partial H}{\partial y} - 3H\frac{\partial r}{\partial y} = 0. \]

The reckoning shows that the above equations for the functions \( \psi(x, y) \) and \( \varphi(x, y) \) are in involution. Namely, all mixed fourth-order derivatives found from different equations are equal. Eliminating \( W \) from the above relations, one arrives at the linearization conditions summarized in Section 2.2. For example, using the expressions for \( C_1 \) and \( C_2 \) given in (4.15) one can find the first derivatives of \( W \):

\[ W_y = \frac{1}{3} WC_2, \quad W_x = \frac{1}{3} W(C_1 - rC_2 + 6r_y). \]

Equating the mixed derivatives \( W_{xy} \) and \( W_{yx} \) one obtains (2.16):

\[ (C_2)_x - (C_1)_y + C_2r_y + r(C_2)_y - 6r_{yy} = 0. \]

Other equations from Section 2.2 are obtained by invoking the equations (4.24) in the expressions for the functions \( D_0, D_1, D_2, D_3, (D_4)_x, \) and \( (D_4)_y \). This completes the proof of Theorem 2.2.

5 Examples to linearization theorems

5.1 Examples on Theorem 2.1

Example 5.1. The equation

\[ y'''' - \left( \frac{6}{y} y' + \frac{3}{x} \right) y'' + \frac{6}{y^2} y'^3 + \frac{6}{xy} y'^2 + \frac{6}{x^2} y' + \frac{6}{x^3} y = 0 \]

(5.1)
is an equation of the form (2.2) with the coefficients
\[ A_1 = -\frac{6}{y}, \; A_0 = -\frac{3}{x}, \; B_3 = \frac{6}{y^2}, \; B_2 = \frac{6}{xy}, \; B_1 = \frac{6}{x^2}, \; B_0 = \frac{6y}{x^3}. \] (5.2)

One can readily verify that the coefficients (5.2) obey the conditions (2.3)-(2.5). We have
\[ 3B_1 - A_0^2 - 3A_0x = 0, \] (5.3)
and Equation (2.6) is written
\[ 2\frac{d\chi}{dx} - \chi^2 = 0. \]

Let us take its simplest solution \( \chi = 0 \). Then, invoking (2.7), we let \( \varphi = x \). Now the equations (2.8) are written
\[
\frac{\partial \ln |\psi_y|}{\partial y} = -\frac{2}{y}, \quad \frac{\partial \ln |\psi_y|}{\partial x} = -\frac{1}{x}
\]
and yield
\[ \psi_y = \frac{K}{xy^2}, \quad K = \text{const}. \]

Hence
\[ \psi = -\frac{K}{xy} + f(x). \]

One can take any particular solution. We set \( K = -1, f(x) = 0 \) and take
\[ \psi = \frac{1}{xy}. \]

Invoking (5.3) and noting that (2.10) yields \( \Omega = 0 \), one can readily verify that the function \( \psi = 1/(xy) \) solves the equation (2.9) as well. Since \( \Omega = 0 \), Equation (2.11) gives \( \alpha = 0 \). Hence, the transformation
\[ t = x, \quad u = \frac{1}{xy} \] (5.4)
maps Equation (5.1) to the linear equation
\[ u'' = 0. \]

**Example 5.2.** Consider the following equation of the form (2.2):
\[ y''' + \frac{3}{y} y'y'' - 3y'' - \frac{3}{y} y'^2 + 2y' - y = 0. \] (5.5)
One can readily verify that its coefficients

\[ A_1 = \frac{3}{y}, \quad A_0 = -3, \quad B_3 = 0, \quad B_2 = -\frac{3}{y}, \quad B_1 = 2, \quad B_0 = -y \]

obey the linearization conditions (2.3)-(2.5). Furthermore,

\[ 3B_1 - A_2^2 - 3A_0x = -3 \]

and Equation (2.6) is written

\[ 6 \frac{d\chi}{dx} - 3\chi^2 = -3. \]

We take its evident solution \( \chi = 1 \) and obtain from (2.7) the equation \( \varphi'' = \varphi' \), whence

\[ \varphi = e^x. \]

The equations (2.8) have the form

\[ \frac{\partial \ln |\psi|}{\partial y} = \frac{1}{y}, \quad \psi_{xy} = 0 \]

and can be readily solved. We take the simplest solution \( \psi = y^2 \) and obtain the following change of variables (2.1):

\[ t = e^x, \quad u = y^2. \]  

(5.6)

Substituting \( \Omega = -2 \) and \( \varphi_x = e^x = t \) in Equation (2.11), we obtain \( \alpha(t) = -2t^{-3} \).

Thus, Equation (5.5) is mapped by the transformation (5.6) to the linear equation

\[ u'''' - \frac{2}{t^3} u = 0. \]  

(5.7)

**Remark 5.1.** In the previous examples, while determining the linearizing transformations, the calculations were confined to particular solutions of the equations (2.6)-(2.9). The reason was that, by considering the general solutions to the equations (2.6)-(2.9), we would add only the symmetry and equivalence transformations for the original and linearized equations, respectively.

**Example 5.3.** Consider the equation

\[ y'''' + y^2 = 0. \]

Here

\[ A_0 = 0, \quad A_1 = 0, \quad B_0 = y^2, \quad B_1 = 0, \quad B_2 = 0, \quad B_3 = 0, \]

(5.9)
and hence
\[ \Omega = 2y, \quad \beta = (3B_1 - A_0^2 - 3A_0x)/3 = 0. \]

The equations (2.3)-(2.4) are satisfied, and Equation (2.13) has the form
\[ 6\alpha'' - 7(\alpha')^2 = 0, \]
where \( \alpha' \) is the derivative of the function \( \alpha(t) \) with respect to \( t \). Setting, e.g. \( \alpha = 1 \), one satisfies all conditions of [6] for linearization by a point transformation.

\[ t = \varphi(x), \quad u = \psi(x, y). \]

However, since \( \Omega_y \neq 0 \) condition (2.5) is not satisfied, and hence, Equation (5.8) is not linearizable (cf. Remark 2.3). Indeed, the relations (3.1)-(3.4) between the functions \( \varphi(x) \) and \( \psi(x, y) \) and the coefficients (5.9) have the following form:
\[ \psi_{yy} = 0, \quad \varphi_x \psi_{xy} - \psi_y \varphi_{xx} = 0 \]
\[ 3\varphi_x^2 \psi_x - \varphi_{xxx} \varphi_x \psi_x - 3\varphi_{xx} \varphi_x \psi_{xx} + \varphi_x^2 \psi_{xxx} + \alpha \psi \varphi_x^5 - y^2 \varphi_x^2 \psi_y = 0. \]

The general solution of the first two equations is
\[ \psi(x, y) = k[y\varphi_x(x) + h(x)], \quad k = \text{const}. \]

Then the third equation yields
\[ \alpha = \frac{y^2 - y\gamma + \lambda}{\varphi_x^2(y\varphi_x + h)}, \tag{5.10} \]
where
\[ \gamma = (\varphi_x^{-1}\varphi_{xxxx} + 3\varphi_x^{-3}\varphi_x^3 - 4\varphi_{xxx}\varphi_x^{-2}), \]
\[ \lambda = \varphi_x^{-3}(\varphi_{xxxx}\varphi_x - 3\varphi_x^2)h' + 3\varphi_x^{-2}\varphi_x h'' - \varphi_x^{-1}h''' \]
Accordingly, the expression (5.10) for \( \alpha \) does not satisfy the condition \( \alpha_y = 0 \), and hence Equation (5.8) is not linearizable.

### 5.2 An example on Theorem 2.2

**Example 5.4.** Consider the non-linear equation
\[ y''' + \frac{1}{y'} \left[ -3y''^2 - xy'^5 \right] = 0. \tag{5.11} \]

It has the form (1.19) with the following coefficients:
\[ r = 0, \quad C_0 = C_1 = C_2 = 0, \]
\[ D_0 = D_1 = D_2 = D_3 = D_4 = 0, \quad D_5 = -x. \tag{5.12} \]
Let us test the equation (5.11) for linearization by using Theorem 2.2. It is manifest that the equations (2.14)-(2.20) are satisfied by the coefficients (5.12). Furthermore, (2.21) also holds since (2.22) yields

$$H = 2.$$  \hfill (5.13)

Thus, the equation (5.11) is linearizable, and we can proceed further. The equations (2.26) are written

$$\frac{\partial W}{\partial x} = 0, \quad \frac{\partial W}{\partial y} = 0$$

and yield $W = \text{const}$. Therefore, the equations (2.23) have the form

$$\frac{\partial \varphi}{\partial x} = 0, \quad \frac{\partial \psi}{\partial x} = -W \frac{\partial \varphi}{\partial y}$$

and hence:

$$\varphi = \varphi(y), \quad \psi = -Wx \varphi'(y) + \omega(y).$$  \hfill (5.14)

Now the third-order equations (2.24) and (2.25) yield the ordinary differential equation

$$\varphi''' = \frac{3 \varphi''^2}{\varphi'}$$  \hfill (5.15)

for $\varphi(y)$ and the partial differential equation

$$\frac{\partial^3 \psi}{\partial y^3} = 3\frac{\varphi''}{\varphi'} \frac{\partial^2 \psi}{\partial y^2} - \frac{3 \varphi''^2}{\varphi'^2} \frac{\partial \psi}{\partial y} - \psi - Wx \varphi'$$  \hfill (5.16)

for $\psi(x, y)$, respectively. Using the expression for $\psi$ given in (5.14) and the equation (5.15) for $\varphi$, we reduce equation (5.16) to

$$3\frac{\varphi''}{\varphi'} \omega'' - \frac{3 \varphi''^2}{2 \varphi'^2} \omega' - \omega - \omega'' = 0.$$

Hence, one can satisfy equation (5.16) by letting $\omega(y) = 0$. Then the construction of the linearizing transformation requires integration of equation (5.15) known in the literature as the Schwarzian equation. Its general solution is provided by the straight lines

$$\varphi = ky + l, \quad k, l = \text{const}.,$$  \hfill (5.17)

and the hyperbolas (see, e.g. [4], p. 8):

$$\varphi = a + \frac{1}{b - cy}, \quad a, b, c = \text{const}.$$  \hfill (5.18)
Let us take the simplest solution \( \varphi = y \) of the form (5.17). Then (2.27) yields \( \alpha = 1 \). Now we set \( W = -1, \omega = 0 \) in (5.14) and arrive at the change of variables
\[
t = y, \quad u = x, \tag{5.19}
\]
reducing (5.11) to the following linear equation:
\[
u'' + u = 0. \tag{5.20}
\]

If one takes the solution to equation (5.15) in the form (5.18), one obtains from (2.27):
\[
\alpha = \frac{(b - cy)^6}{c^3}. \tag{5.21}
\]

Whence, upon eliminating \( b - cy \) by using the equation (5.18) written as
\[
t = a + \frac{1}{b - cy}
\]
one obtains
\[
\alpha(t) = \frac{1}{c^3(t - a)^6}. \tag{5.22}
\]

Hence, the change of variables
\[
t = a + \frac{1}{b - cy}, \quad u = x \tag{5.23}
\]
maps equation (5.11) to the following alternative linear equation:
\[
u'' + \frac{u}{c^3(t - a)^6} = 0. \tag{5.24}
\]

It is known, however, that the two equations (5.20) and (5.22) are equivalent ([4], p. 260). Therefore, the equation (5.20) and the change of variables (5.19) can be regarded as a standard linearization of the equation (5.11).
Bibliography


Group analysis of stochastic differential systems: Approximate symmetries and conservation laws

NAIL H. IBragimov
ALGA, Blekinge Institute of Technology
SE-371 79 Karlskrona, Sweden

GAZANFER ÜNAL
Faculty of Sciences, Istanbul Technical University
Maslak, 80626, Istanbul, Turkey

CLAES JOGRÉUS
ALGA, Blekinge Institute of Technology
SE-371 79 Karlskrona, Sweden

(Received 24 May 2004; accepted 18 June 2004)

Abstract. Approximate symmetries and conservation laws for deterministic and stochastic differential equations with a small parameter are discussed in detail. Determining equations for infinitesimal approximate symmetries of Itô and Stratanovich dynamical systems are derived. It is shown how to derive conserved quantities for stochastic dynamical systems using their approximate symmetries without recourse to Noether’s theorem.

Keywords: Approximate transformations, stochastic dynamical systems, approximate symmetries and conserved quantities for deterministic equations and for Itô and Stratanovich dynamical systems.

1 Introduction

There has been a growing interest in the literature to extend the Lie’s theorems to stochastic differential equations (see for instance [1], [2], [3], [4], [5] and [6]).

A restricted definition of symmetry for Stratonovich dynamical systems was given in [1]. Furthermore, a theorem on the construction of conserved quantities from the symmetries has been given. A more general definition of symmetry for Stratonovich dynamical systems has appeared in [2]. However, the conserved quantity formula given in that works involves the symmetries of the conserved quantities. Yet, it has been shown that symmetries of conserved quantities form a Lie algebra.
In [3], the authors gave the definition of so-called projectable symmetries for Itô dynamical systems. Furthermore, they looked for correspondence between the Lie point symmetries of Fokker-Planck equation and projectable symmetries. A more general definition for Itô systems is given in [4]. However, the latter definition leads to a stochastic determining system which contradicts the result obtained in this work.

Here we also consider the most general form of the transformations of the dependent and independent variables. The symmetry definition given in [5] maps Itô differential to another Itô differential. Therefore, determining systems for the symmetries of the Itô and Stratonovich stochastic dynamical systems become deterministic (i.e. no Wiener terms appear). Symmetries of Stratonovich systems form a Lie algebra. However, for the Itô systems this is not always the case. Furthermore, we show that symmetries of the Itô systems form a subalgebra of the Lie algebra obtained from the Fokker-Planck (forward Kolmogorov) equation. While investigating this correspondence we noticed that conserved quantities can be obtained from the symmetries without resorting to Noether’s theorem. Finally, the results have been illustrated on an application problem.

Consider a one-parameter family of curves

\[ z = \phi_\varepsilon(t) \]

parametrized by \( \varepsilon \).

If \( \varepsilon \) is a small continuous (deterministic) parameter, then the above family of curves defines what is called in [7] an approximate transformation. It is usually written, by replacing \( t \) by \( a \), in the form

\[ \mathbf{x} = \phi(x, a, \varepsilon) \]

where \( x \) denotes the transformed point.

If one replaces the deterministic parameter \( \varepsilon \) by a random variable \( \omega \), one obtains what is called a stochastic process (sometimes termed also a random process). It is written in the form

\[ X_\omega(t). \]

Different stochastic processes are distinguished by choosing \( \omega \) from different probability spaces.

Approximate symmetry analysis of deterministic differential equations with a small parameter has been developed by Baikov, Gazizov and Ibragimov [8], [9]. It has been extended in [10] by incorporating resonances occurring in deterministic dynamical systems. Approximate symmetries of stochastic dynamical systems have not been studied in the literature. We investigate this problem in the sequel (see also [11]).
2 Approximate symmetries and conservation laws of deterministic differential equations

The concept of approximate transformation groups was suggested in [7] for investigating symmetries of differential equations with a small parameter. Subsequently, the new direction of the symmetry analysis of differential equations has been further developed in [8], [9] and applied to variety of nonlinear differential equations (see, e.g. [12]). We outline here the main notions of the theory of approximate symmetries in the first order of precision.

2.1 Approximate transformations

Recall that the notation \( \phi(x, \varepsilon) = o(\varepsilon) \) means that

\[
\lim_{\varepsilon \to 0} \frac{f(x, \varepsilon)}{\varepsilon} = 0.
\]

Consequently, the approximate equation \( f(x, \varepsilon) = g(x, \varepsilon) + o(\varepsilon) \), or briefly \( f \approx g \) means that

\[
f(x, \varepsilon) - g(x, \varepsilon) = o(\varepsilon).
\]

Given a function \( f(x, a, \varepsilon) \), we will choose the standard representative of the class of all functions \( g(x, a, \varepsilon) \approx f(x, a, \varepsilon) \) in the form

\[
 f_0(x, a) + \varepsilon f_1(x, a). \tag{2.1}
\]

**Remark 2.1.** In the general theory, approximate groups are considered with an accuracy of an arbitrary order \( p \geq 1 \) when \( f \approx g \) means that

\[
f(x, \varepsilon) - g(x, \varepsilon) = o(\varepsilon^p).
\]

We use a simplified approach by restricting the theory to the first order of precision, i.e. by assuming \( p = 1 \).

**Definition 2.1.** An approximate transformation

\[
\tau^i \approx f_0^i(x, a) + \varepsilon f_1^i(x, a), \quad i = 1, \ldots, n, \tag{2.2}
\]

is the set of all invertible transformations

\[
\tau^i = f^i(x, a, \varepsilon) \tag{2.3}
\]

such that

\[
f^i(x, a, \varepsilon) \approx f_0^i(x, a) + \varepsilon f_1^i(x, a). \tag{2.4}
\]
It is assumed that the functions $f^i_0(x, a)$ and $f^i_1(x, a)$ are defined and regular in a neighborhood of $a = 0$ and that, in this neighborhood,

$$f^i_0(x, a) = x^i, \quad f^i_1(x, a) = 0$$

if and only if $a = 0$.

**Definition 2.2.** We say that the equations (2.2) define a one-parameter approximate transformation group if any representation (2.3) of (2.2) satisfies the group property (with an error $o(\varepsilon)$):

$$f(f(x, a, \varepsilon), b, \varepsilon) \approx f(x, c, \varepsilon), \quad c = \phi(a, b). \quad (2.5)$$

Upon introducing the canonical parameter $a$, the group property (2.5) can be written in the form

$$f(f(x, a, \varepsilon), b, \varepsilon) \approx f(x, a + b, \varepsilon). \quad (2.6)$$

In this definition, unlike the usual group property, $f$ does not necessarily denote the same function at each occurrence. Specifically, the approximate group property can be equivalently written in the form

$$f(g(x, a, \varepsilon), b, \varepsilon) \approx h(x, a + b, \varepsilon) \quad (2.7)$$

with any functions $f \approx g \approx h$.

**Example 2.1.** Let us consider the one-dimensional case ($n = 1$) and consider the functions

$$f(x, a, \varepsilon) = x + a \left(1 + \varepsilon x + \frac{1}{2} \varepsilon a\right)$$

and

$$g(x, a, \varepsilon) = x + a(1 + \varepsilon x) \left(1 + \frac{1}{2} \varepsilon a\right).$$

The function $f$ has the standard form (2.1), $f = f_0(x, a) + \varepsilon f_1(x, a)$ with

$$f_0(x, a) = x + a, \quad f_1(x, a) = ax + \frac{a^2}{2}.$$

Furthermore, the functions $g$ and $f$ are approximately equal in the first order of precision since

$$g(x, a, \varepsilon) = f(x, a, \varepsilon) + \varepsilon^2 \frac{xa^2}{2}.$$
They form an approximate transformation group. Indeed, the equations
\[ f(f(x, a, \varepsilon), b, \varepsilon) = f(x, a + b, \varepsilon) + \frac{ab(2x + a)}{2} \varepsilon^2 \]
and
\[ f(g(x, a, \varepsilon), b, \varepsilon) = f(x, a + b, \varepsilon) + \frac{a(ax + ab + 2bx + \varepsilon abx)}{2} \varepsilon^2 \]
provide two different, but equivalent representations of the approximate group property (2.7).

The generator of an approximate transformation group (2.2) is the set of all first-order linear differential operators
\[ X = \xi^i(x, \varepsilon) \frac{\partial}{\partial x^i} \]
such that \( \xi^i(x, \varepsilon) \approx \xi^i_0(x) + \varepsilon \xi^i_1(x), \)
where
\[ \xi^i_0(x) = \frac{\partial f^0(x, a)}{\partial a} \bigg|_{a=0}, \quad \xi^i_1(x) = \frac{\partial f^1(x, a)}{\partial a} \bigg|_{a=0}, \quad i = 1, \ldots, n. \]

In calculation, it is convenient to identify \( X \) with its canonical representative:
\[ X = (\xi^i_0(x) + \varepsilon \xi^i_1(x)) \frac{\partial}{\partial x^i}. \] (2.8)

Let
\[ X = X_0 + \varepsilon X_1 \] (2.9)
be a given approximate generator, where
\[ X_0 = \xi^i_0(x) \frac{\partial}{\partial x^i}, \quad X_1 = \xi^i_1(x) \frac{\partial}{\partial x^i}. \]
The corresponding approximate transformation group \( \bar{x} = \bar{x}_0 + \varepsilon \bar{x}_1 \), or in coordinates
\[ \bar{x}^i = \bar{x}^i_0 + \varepsilon \bar{x}^i_1, \quad i = 1, \ldots, n, \]
is determined by the following system of equations:
\[ \frac{d \bar{x}^i_0}{da} = \xi^i_0(\bar{x}_0), \quad \bar{x}^i_0|_{a=0} = x^i, \] (2.10)
\[ \frac{d \bar{x}^i_1}{da} = \sum_{k=1}^{n} \bar{x}^k_1 \cdot \left[ \frac{\partial \xi^i_0(x)}{\partial x^k} \right]_{x=\bar{x}_0} \xi^k_1(\bar{x}_0), \quad \bar{x}^i_1|_{a=0} = 0. \] (2.11)
called the approximate Lie equations.
Example 2.2. Let \( n = 2 \) and let
\[
X = (1 + \varepsilon x^2) \frac{\partial}{\partial x} + \varepsilon xy \frac{\partial}{\partial y}.
\]
Here \( \xi_0(x, y) = (1, 0) \) and \( \xi_1(x, y) = (x^2, xy) \), and the approximate Lie equations (2.10)–(2.11) are written:
\[
\begin{align*}
\frac{d \pi_0}{da} &= 1, & \frac{d \gamma_0}{da} &= 0, & \pi_0|_{a=0} &= x, & \gamma_0|_{a=0} &= y, \\
\frac{d \pi_1}{da} &= (\pi_0)^2, & \frac{d \gamma_1}{da} &= \pi_0 \gamma_0, & \pi_1|_{a=0} &= 0, & \gamma_1|_{a=0} &= 0.
\end{align*}
\]
Integration yields:
\[
\begin{align*}
\pi &\approx x + a + \varepsilon \left( ax^2 + a^2 x + \frac{a^3}{3} \right), \\
\gamma &\approx y + \varepsilon \left( axy + \frac{a^2}{2} y \right).
\end{align*}
\]

2.2 Approximate symmetries

We denote by \( z \equiv (z^1, \ldots, z^N) = (x, u, u_{(1)}, \ldots, u_{(k)}) \) the set of independent variables \( x = (x^1, \ldots, x^n) \) and dependent variables \( x = (u^1, \ldots, u^m) \) together with the partial derivatives \( u_{(1)}, \ldots, u_{(k)} \) of \( u \) with respect to \( x \) of the respective orders \( 1, \ldots, k \).

Consider an approximate differential equation of order \( k \):
\[
F(z, \varepsilon) \equiv F_0(z) + \varepsilon F_1(z) \approx 0.
\]
We include here also the systems of equations assuming that \( F_0 \) and \( F_1 \) can be vector valued functions. Let \( G \) be a one-parameter approximate transformation group and let its prolongation to the the derivatives involved in equation (2.12) have the form:
\[
\begin{align*}
\pi^i &\approx f(z, a, \varepsilon) \equiv f_0^i(z, a) + \varepsilon f_1^i(z, a), & i &= 1, \ldots, N.
\end{align*}
\]

Definition 2.3. We say that equation (2.12) is \textit{approximately invariant} with respect to group \( G \) if
\[
F(f(z, a, \varepsilon), \varepsilon) = o(\varepsilon)
\]
whenever \( z = (z^1, \ldots, z^N) \) satisfies equation (2.12).

Consider an approximate transformation group of the independent and dependent variables with the generator
\[
X = X^0 + \varepsilon X^1,
\]
where
\[
X^0 = \xi_0^i(x, u) \frac{\partial}{\partial x^i} + \eta_0^a(x, u) \frac{\partial}{\partial u^a}, & \quad X^1 = \xi_1^i(x, u) \frac{\partial}{\partial x^i} + \eta_1^a(x, u) \frac{\partial}{\partial u^a}.
\]
We will write the prolongation of \( X \) to the derivatives involved in the equation (2.12) in the following form:

\[
\tilde{X} = \tilde{X}^0 + \varepsilon \tilde{X}^1 = \zeta_k^0(z) \frac{\partial}{\partial z_k} + \varepsilon \zeta_k^1(z) \frac{\partial}{\partial z_k}.
\] (2.16)

**Theorem 2.1.** The equation (2.12) is approximately invariant under the approximate transformation group with the generator (2.14) if and only if

\[
\left[ \tilde{X}^0 F_0(z) + \varepsilon \left( \tilde{X}^1 F_0(z) + \tilde{X}^0 F_1(z) \right) \right]_{(2.12)} = o(\varepsilon).
\] (2.17)

Equation (2.17) is the determining equation for infinitesimal approximate symmetries. The determining equation (2.17) can be written in the form

\[
\tilde{X}^0 F_0(z) = \lambda(z) F_0(z),
\] (2.18)

\[
\tilde{X}^1 F_0(z) + \tilde{X}^0 F_1(z) = \lambda(z) F_1(z).
\] (2.19)

The factor \( \lambda(z) \) is determined by (2.18) and then substituted into (2.19). The latter equation must hold for all solutions of \( F_0(z) = 0 \).

**Remark 2.2.** It follows from Eq. 2.18 that if \( X = X^0 + \varepsilon X^1 \) is an approximate symmetry with \( X^0 \neq 0 \), then the operator

\[
X^0 = \xi_0(x, u) \frac{\partial}{\partial x^i} + \eta_0(x, u) \frac{\partial}{\partial u^i}
\] (2.20)

is an exact symmetry for the unperturbed equation

\[ F_0(z) = 0. \]

The corresponding approximate symmetry generator \( X = X^0 + \varepsilon X^1 \) for the perturbed equation (2.12) is called a deformation of the infinitesimal symmetry \( X^0 \) of the equation (2.20) caused by the perturbation \( \varepsilon F_1(z) \).

**Definition 2.4.** The symmetry \( X^0 \) of the unperturbed equation (2.20) is called a stable symmetry if there exists \( X^1 \) such that \( X = X^0 + \varepsilon X^1 \) is an approximate symmetry for the perturbed equation (2.12). In particular, if the most general symmetry Lie algebra of the equation (2.20) is stable, we say that the perturbed equation (2.12) inherits the symmetries of the unperturbed equation.

**Remark 2.3.** The determining Equations (2.18) - (2.19) are equivalent to the following equations:

\[
\left. \tilde{X}^0 F_0(z) \right|_{F_0(z)=0} = 0.
\] (2.21)
where $H$ is defined by

$$H = \frac{1}{\varepsilon} \left[ X^0 \left( F_0(z) + \varepsilon F_1(z) \right) \right]_{F_0(z) + \varepsilon F_1(z) = 0}.$$

(2.23)

**Example 2.3.** In [13], the nonlinear wave equations with a small dissipation $\varepsilon u_t :$

$$u_{tt} - (f(u)u_x)_x + \varepsilon \varphi(u)u_t = 0$$

are classified according to their approximate symmetries. We will illustrate the algorithm of calculation of approximate symmetries by considering the following particular case of the above equations:

$$u_{tt} + \varepsilon u_t = (u^\sigma u_x)_x, \quad \sigma \neq 0.$$

(2.24)

An approximate group generator is written

$$X = X^0 + \varepsilon X^1 \equiv (\tau_0 + \varepsilon \tau_1) \frac{\partial}{\partial t} + (\xi_0 + \varepsilon \xi_1) \frac{\partial}{\partial x} + (\eta_0 + \varepsilon \eta_1) \frac{\partial}{\partial u},$$

where $\tau_\nu, \xi_\nu,$ and $\eta_\nu (\nu = 0, 1)$ are unknown functions of $t, x,$ and $u.$

At first we have to calculate the exact symmetries of the unperturbed equation

$$u_{tt} = (u^\sigma u_x)_x.$$

(2.25)

Solution of the determining equation (2.21) yields

$$X^0 = (C_1 + C_3 u) \frac{\partial}{\partial t} + (C_2 + C_3 x + C_4 x) \frac{\partial}{\partial x} + C_4 \frac{2u}{\sigma} \frac{\partial}{\partial u}.$$

(2.26)

Hence, the exact symmetries of Eq. (2.25) form the four-dimensional Lie algebra spanned by

$$X^0_1 = \frac{\partial}{\partial t}, \quad X^0_2 = \frac{\partial}{\partial x}, \quad X^0_3 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \quad X^0_4 = x \frac{\partial}{\partial x} + \frac{2u}{\sigma} \frac{\partial}{\partial u}. $$

(2.27)

Substitution of the generator $X^0$ into (2.26) in (2.23) gives

$$H = C_3 u_t.$$

Consequently, the determining equation (2.22) is written:

$$\tilde{X}^1 \left( u_{tt} - u^\sigma u_{xx} + \sigma u^{\sigma-1} u_x^2 \right) \bigg|_{(9.87)} + C_3 u_t = 0,$$

(2.28)
where $\tilde{X}^1$ is the prolongation of the operator

$$X^1 = \tau_1 \frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial u}.$$  

Upon setting $u_{tt} = (u^\sigma u_x)_x$, the left-hand side of Eq. (2.28) becomes a polynomial in the variables $u_{tx}, u_{xx}, u_t, u_x$. Equating to zero its coefficients, we obtain:

$$\tau_1 = \tau_1(t), \quad \tau_1''(t) = 0; \quad \xi_1 = \xi_1(x), \quad \xi_1''(x) = 0; \quad \eta_1 = \frac{2u}{\sigma} [\xi_1'(x) - \tau_1'(t)];$$

$$\left(\sigma + \frac{4}{3}\right) \xi_1'' = 0, \quad \left(\frac{4}{\sigma} + 1\right) \tau_1'' = C_3.$$  

The solution of this system can be written in the form:

$$\tau_1 = C_3 \frac{\sigma t^2}{2(\sigma + 4)} + A_1 + A_3 t, \quad \xi_1 = A_2 + A_3 x + A_4 x, \quad \eta_1 = \frac{2u}{\sigma} \left( A_4 - \frac{\sigma}{\sigma + 4} C_3 t \right).$$  

Thus, equation (2.24) with an arbitrary constant $\sigma \neq 0$ has the following infinitesimal approximate symmetries:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \left( t + \frac{\varepsilon \sigma t^2}{2(\sigma + 4)} \right) \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - \frac{2\varepsilon tu}{\sigma + 4} \frac{\partial}{\partial u},$$

$$X_4 = x \frac{\partial}{\partial x} + \frac{2u}{\sigma} \frac{\partial}{\partial t}, \quad X_5 = \varepsilon X_1, \quad X_6 = \varepsilon X_2, \quad X_7 = \varepsilon X_4, \quad X_8 = \varepsilon \left( t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} \right) \approx \varepsilon X_3.$$  

They span an eight-dimensional approximate Lie algebra and generate an eight-parameter approximate transformations group. The operators $X_1$ to $X_4$ show that the symmetry Lie algebra $L_4$ of equation (2.25) is stable. Hence, the perturbed equation (2.24) inherits all symmetries of the unperturbed equation (2.25).

It is transparent from the last two equations (2.29) that two numerical values of the exponent of nonlinearity, $\sigma = -4/3$ and $\sigma = -4$, have a particular significance (see [14], Sec 9.5.3).

In the case $\sigma = -4/3$, the corresponding unperturbed equation (2.25), i.e. the equation

$$u_{tt} = (u^{-4/3} u_x)_x$$

admits, along with (2.27), the additional operator

$$X_5^0 = x^2 \frac{\partial}{\partial x} - 3xu \frac{\partial}{\partial u}.$$  

The perturbed equation

$$u_{tt} + \varepsilon u_t = (u^{-4/3} u_x)_x$$
admits the 10-dimensional approximate Lie algebra $L_{10}$ spanned by

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = (t - \frac{\varepsilon t^2}{4}) \frac{\partial}{\partial t} + \frac{x}{4} \frac{\partial}{\partial x} - \frac{3}{4} \varepsilon t u \frac{\partial}{\partial u},$$

$$X_4 = \frac{x}{4} \frac{\partial}{\partial x} - \frac{3}{2} u \frac{\partial}{\partial u}, \quad X_5 = x^2 \frac{\partial}{\partial x} - 3 xu \frac{\partial}{\partial u}.$$

In this case, the maximal symmetry algebra $L_5$ is also stable, i.e., the perturbed equation inherits all symmetries of the unperturbed equation.

In the case $\sigma = -4$, the corresponding unperturbed equation (2.25), i.e. the equation

$$u_{tt} = (u^{-4} u_x)_x$$

admits, along with (2.27), the additional operator

$$X^0_5 = t u \frac{\partial}{\partial u}.$$

The perturbed equation

$$u_{tt} + \varepsilon u_t = (u^{-4} u_x)_x$$

has the following approximate symmetries:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \varepsilon \left( t \frac{\partial}{\partial t} + \frac{x}{4} \frac{\partial}{\partial x} \right), \quad X_4 = \frac{x}{4} \frac{\partial}{\partial x} - \frac{3}{2} u \frac{\partial}{\partial u},$$

$$X_5 = \varepsilon X_1, \quad X_6 = \varepsilon X_2, \quad X_7 = \varepsilon X_4, \quad X_8 = \varepsilon \left( t^2 \frac{\partial}{\partial t} + tu \frac{\partial}{\partial u} \right).$$

It follows that the operators

$$X^0_3 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} \quad \text{and} \quad X^0_5 = t^2 \frac{\partial}{\partial t} + tu \frac{\partial}{\partial u}$$

of the symmetry Lie algebra $L_5$ of the unperturbed equation are unstable. Hence, in this example the perturbed equation does not inherit all symmetries of the unperturbed equation.

### 2.3 Approximate version of Noether’s theorem

According to [15], one can extend the Noether theorem to approximate symmetries and obtain approximate conservation laws for Euler-Lagrange type equations with a small parameter that admit an approximate transformation group. Namely, the following statement can be proved merely by replacing in the proof of Noether’s theorem exact equations by the corresponding approximate equations.
Theorem 2.2. Suppose that the Euler-Lagrange equation
\[ \frac{\delta L}{\delta u^\alpha} \equiv \frac{\partial L}{\partial u^\alpha} - D_i \left( \frac{\partial L}{\partial u_i^\alpha} \right) = 0 \] (2.30)
is approximately invariant with respect to the approximate transformation group with a generator
\[ X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \]
and that
\[ \tilde{X}L + LD_i(\xi^i) = D_i(B^i) + o(\varepsilon) \] (2.31)
with some functions \( B^i(x, u, \ldots) \). Then the vector
\[ C^i = L\xi^i + (\eta^\alpha - \xi^i u_j^\alpha) \frac{\partial L}{\partial u_i^\alpha} - B^i \] (2.32)
satisfies the following conservation law for the equations (2.30):
\[ D_i(C^i) = o(\varepsilon). \] (2.33)

Example 2.4. The perturbed wave equation
\[ u_{tt} - u_{xx} - u_{yy} + \varepsilon u_t = 0 \] (2.34)
has the Lagrangian
\[ L = \frac{1}{2} e^{\varepsilon t}(u_t^2 - u_x^2 - u_y^2) \]
and admits, as one of its approximate symmetries, the following operator:
\[ X = (t^2 + x^2 + y^2) \frac{\partial}{\partial t} + 2tx \frac{\partial}{\partial x} + 2ty \frac{\partial}{\partial y} - \left( t + \frac{\varepsilon}{2}(t^2 + x^2 + y^2) \right) u \frac{\partial}{\partial u}. \]
The condition (2.31) holds with the functions
\[ B^1 \approx -(1 + \varepsilon t)u^2/2, \quad B^2 \approx \varepsilon xu^2/2, \quad B^3 \approx \varepsilon yu^2/2. \]
Formula (2.32) yields the following approximate conservation law:
\[ D_t(e^{\varepsilon t}[-\frac{s^2}{2}(u_t^2 + u_x^2 + u_y^2)] - u_t(tu + \frac{\varepsilon}{2}s^2 u + 2txu_x + 2tyu_y)) \]  
\[ + D_x(e^{\varepsilon t}[t(x(u_t^2 + u_x^2 - u_y^2)) + u_x(tu + \frac{\varepsilon}{2}s^2 u + s^2 u_t - \frac{\varepsilon}{2}xu^2 + 2tyu_y)]) \]
\[ + D_y(e^{\varepsilon t}[ty(u_t^2 - u_x^2 + u_y^2)] + u_y(tu + \frac{\varepsilon}{2}s^2 u + s^2 u_t - \frac{\varepsilon}{2}yu^2 + 2txu_x)]) = o(\varepsilon), \]
where \( s^2 = t^2 + x^2 + y^2. \)
3 Stochastic differential equations

3.1 Stochastic processes

A random variable is a variable which attains values in a set $\Omega$ with certain probabilities. Discrete random variables can attain only some fixed values, while continuous variables can have any value on a certain interval. Below, only continuous random variables are considered. The probability $P(X \leq x)$ is denoted $F(x)$ (the cumulative distribution function, cdf). The derivative $F'(x) \equiv p(x)$ and is called the probability function or the density. Thus, the probability to get a value between $x$ and $x + dx$ is approximately $p(x)dx$.

A stochastic process is a family of random variables $X(t, \omega)$ assuming values in $\mathbb{R}^n$ and parametrized by the time parameter $t \in T$ and a random parameter $\omega$ to denote that the value of the process at a certain time $t$ is random. If $T = \{1, 2, \ldots\}$, $X = X(t, \omega)$ is said to be a process in discrete time (i.e. a sequence of random variables). If $T$ is an interval in $\mathbb{R}$, e.g. $[0, \infty)$, $X$ is said to be a stochastic process in continuous time. The process for a fixed value of $\omega$ is called a sample path (or a realisation) of the process. On the other hand, if $t$ is fixed, the value of $X$ is a random variable. In the continuous time case, the stochastic process resembles a deterministic function, but the value in each point is random. However, conditions are often added to make the sample path of the process continuous. Below, $\omega$ will be omitted to simplify notations. In the figure below, two sample paths of the same process are shown, i.e. the characteristics like e.g. the density $p(x)$ are the same, but $\omega$ is different, corresponding to e.g. two various simulations. In general, processes are either called discrete or continuous. A discrete stochastic process can only attain some fixed values, while continuous stochastic processes can attain any value on an interval. Note that a continuous process does not necessarily have continuous sample paths.

**Definition 3.1.** A stochastic process $B(t)$ is called a Wiener process (Brownian motion) if the following conditions are satisfied:

1) $B(0) = 0$ with probability one.

2) For $0 \leq s \leq t$,

\[ B(t) - B(s) \in N(0, \sqrt{t-s}), \]

i.e. the process has a Gaussian distribution with mean zero and variance equal to the time difference.

3) With probability one (almost surely), the paths of $B$ are continuous.

4) The increments of $B$ are independent for disjunct time intervals.

A white noise in discrete time is simply a sequence of independent random variables. In continuous time, the value of the white noise at a point $t$ is a random variable
which is independent of the value at all other times no matter how small the time difference is. Equivalently, white noise is defined as a random process which has an autocorrelation function which is a Dirac function. As a result of this independence, the so-called spectral density is constant, i.e. all frequencies are given the same weight. White noise is consequently discontinuous almost surely, i.e. if a time \( t \) is selected randomly, the probability that the process is continuous in this point equals zero. The background and derivation of these and related results are given in [16]. Integration of Gaussian white noise with respect to time gives Brownian motion (Wiener process). Consequently, the Wiener process is continuous but non-differentiable almost surely.

Let \( B(t) \) be a Wiener process as above and let \( x(t) \) be the solution of the differential equation

\[
dx = f(x, t)dt + b(x, t)dB(t), \quad x(0) = c. \tag{3.1}
\]

or equivalently

\[
x(t) = c + \int_0^t f(x, u)du + \int_0^t b(x, u)dB(t) \tag{3.2}
\]

The differential \( dB \) is often not interpreted, but (3.1) is seen as an easy way to write the more rigorously defined equation (3.2), where the stochastic integration is explained below. Here \( c \) is a constant or a random variable with finite variance, while \( f \) and \( b \) are continuous deterministic functions which satisfy the Lipschitz condition

\[
|f(y, t) - f(x, t)| + |b(y, t) - b(x, t)| \leq C |y - x|
\]

and the growth condition

\[
|f(x, t)| + |b(x, t)| \leq C(1 + |x|).
\]

Then a unique solution exists, i.e. a random process \( x(t) \) satisfying (3.1). Such a solution is called an Itô process.

### 3.2 The Itô and Stratonovich integrals

Random integrals like in (3.2) can be defined in several ways. In the stochastic Riemann sum

\[
\sum f(t_j^*) \chi_{t_j, t_{j-1}}
\]

it matters how the point \( t_j^* \) is chosen - unlike the deterministic case. If \( t_j^* \) is chosen as the left endpoint \( t_j \) the resulting integral is the so-called Itô integral

\[
\int f(t, \omega)dB_t(\omega).
\]

If the mean of the interval is used, i.e. \( t_j^* = (t_j + t_{j+1})/2 \), the result is the Stratonovich integral

\[
\int f(t, \omega) \circ dB_t(\omega),
\]
which is the stochastic counterpart to the Lebesgue integral. One interpretation of the integral can be transformed to the other when dealing with the equation (3.1); see e.g. [17]. Applications of the above integrals to engineering problems and modelling by stochastic differential equations can be found, e.g. in [18] and [19].

For calculating differentials the following theorem is essential, see e.g. [17].

**Theorem 3.1.** (The multi-dimensional Itô formula). Let
\[ x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \]
be an \( n \)-dimensional Itô process given by (3.1):
\[ dx = f(x,t)\, dt + b(x,t)\, dB(t), \quad x(0) = c. \]
where \( f \) is a vector and \( b \) is an \( n \times n \) matrix, whose components all satisfy the conditions in (3.1) above. Let \( g(t,x) = (g_1(t,x), \ldots, g_p(t,x)) \) be a \( C^2 \) map from \([0, \infty) \times \mathbb{R}^n \) into \( \mathbb{R}^p \). Then
\[ y(t) = g(t,x(t)) \]
is a \( p \)-dimensional Itô process and the differential equals
\[ dy_i = \frac{\partial g_i}{\partial t} \, dt + \sum_j \frac{\partial g_i}{\partial x_j} \, dx_j + \frac{1}{2} \sum \frac{\partial^2 g_i}{\partial x^2} dx_i \, dx_j. \quad (3.3) \]

Here \( dx_i \, dx_j \) is calculated using the "Itô multiplication table"
\[ dt \cdot dt = 0, \quad dt \cdot dB_i = dB_i \cdot dt = 0, \quad dB_i \cdot dB_j = \delta_{ij} dt, \]
where \( \delta_{ij} \) is the Kronecker delta. For derivation of the theorem, see [17].

**Example 3.1.** Choose \( x(t) = B(t) \), i.e. \( f = 0 \) and \( b = 1 \) in (3.1). Using map \( y(t) = x(t)^2 = B(t)^2 \), the differential \( dy = d(B^2) \) is wanted. Then, by means of Itô’s formula,
\[ dy = \frac{\partial g}{\partial t} \, dt + \frac{\partial g}{\partial x} \, dx + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} (dx)^2 = 0 \, dt + 2BdB + \frac{1}{2} (dx)^2. \]
Since \( (dX)^2 = (dB)^2 = dt \), the result is
\[ dy = d(B^2) = 2BdB + dt = 2xdx + dt. \]

Compared to the case where \( x(t) \) is a deterministic function, an extra term \( dt \) is added. Generally, the differentiation of a function of \( B \) gives one \( dB \) and one \( dt \)-term.

Since the solution \( x(t) \) of (3.1) is a random process, its distribution is described by a probability density \( p(x) \) which is calculated by solving the Fokker-Planck diffusion equation (also known as the forward Kolmogorov equation):
\[ \frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 (b(t,x)p)}{\partial x^2} - \frac{\partial (f(t,x)p)}{\partial x}. \]
4 Approximate symmetries of Itô dynamical systems

Here we consider an Itô dynamical system of the form

$$\frac{dx_i}{dt} = (f^0_i(x, t) + \varepsilon f^1_i(x, t))dt + \sqrt{\varepsilon} c_{i\alpha} dB_{\alpha}, \quad (i = 1, \ldots, n; \alpha = 1, \ldots, r),$$  \hspace{1cm} (4.1)

where $f^0_i(x, t) + \varepsilon f^1_i(x, t)$ is an approximate drift vector, $c_{i\alpha}$ is a constant diffusion matrix, $\varepsilon \ll 1$ is a small positive perturbation parameter and $dB_{\alpha}$ is a vector Wiener process. In what follows summation convention applies to repeated indices hereafter.

We give the definition of the approximate symmetries. This allows us to obtain a deterministic hierarchy of determining system involving random variables.

**Definition 4.1.** If infinitesimal transformations

$$\bar{x}_i \approx x_i + a(\xi^0_i(x, t) + \varepsilon \xi^1_i(x, t)), \quad \bar{t} \approx t + a(\tau^0(x, t) + \varepsilon \tau^1(x, t))$$  \hspace{1cm} (4.2)

leave Eqs. (4.1) and the identities (see [20])

i) $dB_{\alpha}dB_{\beta} = \delta_{\alpha\beta}dt$,  

ii) $dtdB_{\alpha} = 0$,  

iii) $dtdt = 0$,  

approximately form invariant i.e.,

$$d\bar{x}_i = [(f^0_i(\bar{x}, t) + \varepsilon f^1_i(\bar{x}, t))]dt + \sqrt{\varepsilon} c_{i\alpha}(\bar{x}, t) d\bar{B}_{\alpha},$$

$$\hspace{1cm} (i = 1, \ldots, n; \alpha = 1, \ldots, r),$$  \hspace{1cm} (4.4)

then they are called approximate symmetry transformations for the Itô dynamical system. Here $a$ is a group parameter.

Why do we require (4.3) to remain invariant under the transformations (4.2)? To be able to provide an answer to this question, we now consider the evolution of the sufficiently smooth scalar function $I(x, t)$ under the flow of the Itô equations (4.1) i.e., Itô’s formula (see, e.g. [20] and [17])

$$dI(x, t) = I_t dt + I_j dx_j + \frac{1}{2} I_{jk} dx_j dx_k.$$  \hspace{1cm} (4.5)

Substitution of $dx_i$ defined by (4.1) into Eq. (4.5) yields

$$dI = (I_t + (f^0_j + \varepsilon f^1_j)I_j) dt + \sqrt{\varepsilon} I_{j\alpha} c_{j\alpha} dB_{\alpha} + \frac{\varepsilon}{2} I_{j\beta} c_{j\alpha} c_{k\beta} dB_{\alpha} dB_{\beta}$$

$$+ \frac{1}{2} I_{jk} [(f^0_j + \varepsilon f^1_j)(f^0_k + \varepsilon f^1_k) dt] + \sqrt{\varepsilon} (f^0_j + \varepsilon f^1_j) c_{j\alpha} d\bar{B}_{\alpha}$$

$$+ \sqrt{\varepsilon} (f^0_k + \varepsilon f^1_k) c_{k\alpha} d\bar{B}_{\alpha}. $$  \hspace{1cm} (4.6)

\textsuperscript{4}We will use the comma for the partial derivative with respect to the coordinate appearing in the subscript, e.g. $I_t = \partial I/\partial t$. 


Using the identities given in (4.3) in (4.6) leads to Itô differential of the scalar function \( I(x, t) \)
\[
dI = [I_x + f_i^0 I_{x,i} + \varepsilon(f_j I_{x,j} + \frac{1}{2} c_{j\alpha} c_{\alpha\beta} I_{x,j}^m)]dt + \sqrt{\varepsilon} c_{\alpha\beta} I_{x,j} dB_\alpha
\] (4.7)
Form invariance of (4.1) under transformations (4.2) requires that Itô differential of \( \tilde{I}(\tilde{x}, \tilde{t}) \) should read as
\[
d\tilde{I} = [\tilde{I}_x + \tilde{f}_i^0 \tilde{I}_{x,i} + \varepsilon(\tilde{f}_j \tilde{I}_{x,j} + \frac{1}{2} c_{j\alpha} c_{\alpha\beta} \tilde{I}_{x,j}^m)]d\tilde{t} + \tilde{I}_{x,j} \tilde{g}_{\alpha,j} dB_\alpha.
\] (4.8)
For this to happen the identities (4.3) have to remain form invariant under (4.2):
\[
i) \quad d\tilde{B}_\alpha d\tilde{B}_\beta = \delta_{\alpha\beta} d\tilde{t}, \quad ii) \quad d\tilde{t} d\tilde{B}_\alpha = 0, \quad iii) \quad d\tilde{t} d\tilde{t} = 0.
\] (4.9)
This justifies the idea behind the symmetry definition.
We now proceed to seek the determining system for the symmetries of (4.1). To achieve this we have to calculate each term in (4.8) in terms of the original variables \( x \) and \( t \).
\[
\tilde{f}_i^m(x + a(\xi^0 + \varepsilon \xi^1), t + a(\tau^0 + \varepsilon \tau^1))
\]
\[
= f_i^m(x, t) + a[\xi^0_j f_{i,j}^m + \tau^0_i f_{i,t}^m + \varepsilon(\xi^1_j f_{i,j}^m + \tau^1_i f_{i,t}^m)] + O(a^2).
\] (4.10)
The Itô differential of (4.2) reads
\[
i) \quad d\tilde{x}_i = dx_i + a(d\xi_i^0 + \varepsilon d\xi_i^1) + O(a^2),
\]
\[
ii) \quad d\tilde{t} = dt + a(d\tau^0 + \varepsilon d\tau^1) + O(a^2),
\] (4.11)
where,
\[
d\xi_i^m = [\xi_i^m + f_j^0 \xi_{i,j}^m + \varepsilon(f_j^1 \xi_{i,j}^m + \frac{1}{2} c_{j\alpha} c_{\alpha\beta} \xi_{i,j}^m)]dt + \sqrt{\varepsilon} c_{j\alpha} \xi_{i,j}^m dB_\alpha,
\]
\[
d\tau_i^m = [\tau_i^m + f_j^0 \tau_{i,j}^m + \varepsilon(f_j^1 \tau_{i,j}^m + \frac{1}{2} c_{j\alpha} c_{\alpha\beta} \tau_{i,j}^m)]dt + \sqrt{\varepsilon} c_{j\alpha} \tau_{i,j}^m dB_\alpha.
\]
Rendering (4.11ii) back to the right hand side of (4.9i) yields
\[
dB_\alpha dB_\beta = dB_\alpha dB_\beta + a[\tau_i^0 + f_j^0 \tau_{i,j}^0 + \varepsilon(\tau_i^1 + f_j^0 \tau_{i,j}^1) + \varepsilon f_j^1 \tau_{i,j}^0]
\]
\[
+ \frac{1}{2} c_{j\alpha} c_{\alpha\beta} \tau_{i,j}^0] + \varepsilon^{1/2} c_{\gamma} (\tau^0 + \varepsilon \tau^1)_j \frac{dB_\gamma}{dt} dB_\alpha dB_\beta.
\]
Let us first set \( \beta = \alpha \) in above equation and then Taylor expansion leads to
\[
d\tilde{B}_\alpha = dB_\alpha + \frac{1}{2} a[\tau_i^0 + f_j^0 \tau_{i,j}^0 + \varepsilon(\tau_i^1 + f_j^0 \tau_{i,j}^1) + \varepsilon f_j^1 \tau_{i,j}^0]
\]
\[
+ \frac{1}{2} c_{j\alpha} c_{\alpha\beta} \tau_{i,j}^0] + \varepsilon^{1/2} c_{\gamma} (\tau^0 + \varepsilon \tau^1)_j \frac{dB_\gamma}{dt} dB_\alpha.
\] (4.12)
Note, that the infinitesimal transformation law given in (4.12) has been partially captured in [3] by considering so-called projectable symmetries, i.e. by imposing the restriction \( \tau = \tau(t) \), and hence obtaining the similar restriction of the infinitesimal transformations of probability density function. In [4], the authors obtain exactly the same formula but they rely on a theorem given in [17].

We continue our calculations and consider the remaining conditions. Let us consider (4.9 ii). Substituting (4.11 ii) and (4.12) into the left hand side of (4.9 ii) we obtain:

\[
\frac{d \alpha}{t} \frac{d \alpha}{B} = g_{j\alpha} \tau_{,j} dt = 0.
\]

This leads to

\[
c_{j\alpha} \tau_{,j}^0 = 0, \quad c_{j\alpha} \tau_{,j}^1 = 0. \tag{4.13}
\]

Rendering (4.13) back into (4.12) yields

\[
d \alpha \frac{d B}{\alpha} = \frac{1}{2} \left[ f_{0,0} \tau_{,j}^0 + f_{1,0} \tau_{,j}^1 + \varepsilon f_{0,0} \tau_{,j}^1 + f_{1,0} \tau_{,j}^1 + \frac{1}{2} c_{j\alpha} c_{\alpha \beta} \tau_{,j}^0 \right] dB_{\alpha}. \tag{4.14}
\]

Equation given in (4.9 iii) do not introduce new constraints on \( \tau(x, t) \) and \( dB_{\alpha}(t) \).

**Remark 4.1.** Note, that the transformation law (4.14) for the infinitesimal increments of Wiener processes is different from the one given in [4] (namely, cf. (4.14) with the equation (20) in [4]) and it generalizes the result obtained in [3]. Notice also that the first condition (4.13) was imposed in [4] as a sufficient condition to get rid of stochastic terms thus simplifying the determining equations. On the contrary, we show that this condition is not only sufficient but also necessary and that Eqs. (4.13) follow from Eq. (4.3) of the symmetry definition.

We now substitute (4.10), (4.11) and (4.14) into equation given in (4.4) to obtain

\[
dx_i = (f_i^0 + \varepsilon f_i^1) dt + \sqrt{\varepsilon} c_{\alpha \delta} dB_{\alpha} + a \left[ T_i - Z_i + \frac{\sqrt{\varepsilon}}{2} c_{\alpha \delta} Q dB_{\alpha} \right],
\]

where

\[
T_i = \left[ \xi_{i,0}^0 f_{i,j}^0 + \tau_0^0 f_{i,j}^0 + f_{i,j}^0 \tau_0^0 + f_{i,j}^0 \tau_0^1 + \varepsilon (\xi_{i,j}^0 f_{i,j}^0 + \tau_1^0 f_{i,j}^1 + f_{i,j}^0 \tau_1^0) + \frac{1}{2} c_{j\alpha} c_{\alpha \beta} \tau_{,j}^0 \right] dt,
\]

\[
Z_i = \left[ \xi_{i,0}^1 + f_{i,j}^0 \xi_{i,j}^0 + \varepsilon (\xi_{i,j}^1 + f_{i,j}^0 \xi_{i,j}^1 + f_{i,j}^0 \xi_{i,j}^0 + f_{i,j}^1 \xi_{i,j}^0 + \frac{1}{2} c_{j\alpha} c_{\alpha \beta} \xi_{i,j}^0) \right] dt + \sqrt{\varepsilon} (c_{j\alpha} c_{\alpha \beta} \xi_{i,j}^0) dB_{\alpha},
\]

\[
Q = \tau_0^0 + f_{i,j}^0 \tau_0^0 + \varepsilon (\tau_1^0 + f_{i,j}^0 \tau_1^0 + f_{i,j}^0 \tau_0^0 + \frac{1}{2} c_{j\alpha} c_{\alpha \beta} \tau_{,j}^0).
\]

We deduce from these equations the following theorem.
Theorem 4.1. The infinitesimal transformations (4.2) provide approximate symmetries for the Itô equations (4.1) if and only if the infinitesimals \( \xi_i(x, t) \) and \( \tau(x, t) \) satisfy the following determining equations:

\[
\begin{align*}
\xi_{i,t}^0 + f_j^0 \xi_{i,j}^0 - \xi_j^0 f_{i,j}^0 - \tau_j^0 f_{i,t}^0 - f_{i}^0 \tau_j^0 &= 0, \\
c_{j\alpha} \xi_{i,j}^0 - \frac{c_{i\alpha}}{2} (\tau_j^0 + f_j^0 \tau_j^0) &= 0, \tag{4.15} \\
c_{j\alpha} \tau_j^0 &= 0.
\end{align*}
\]

Notice that this system does not involve Wiener terms (i.e., \( dB_{\alpha} \)) and hence, it is deterministic.

5 Approximate symmetries of Stratonovich dynamical systems

The Stratonovich dynamical system with a small parameter \( \varepsilon \) reads as

\[
dx_i = (f_i^0(x, t) + \varepsilon f_i^1(x, t))dt + \sqrt{\varepsilon} c_{i\alpha}(x, t) \circ dB_{\alpha}, \tag{5.1}
\]

where \( i = 1, \ldots, n; \alpha = 1, \ldots, r; \) and \( \circ \) denotes the Stratonovich derivative.

Theorem 5.1. The infinitesimal transformations (4.2) provide approximate symmetries for the Stratonovich dynamical system (5.1) if and only if the infinitesimals \( \xi_i(x, t) \) and \( \tau(x, t) \) satisfy the following determining equations:

\[
\begin{align*}
\xi_{i,t}^1 + f_j^0 \xi_{i,j}^1 - \xi_j^1 f_{i,j}^0 - \tau_j^0 f_{i,t}^0 - f_{i}^0 \tau_j^0 &= -f_j^0 \xi_{i,j}^0 + \xi_j^0 f_{i,j}^1, \\
\tau_{i,t}^1 + f_j^0 \tau_{i,j}^1 - \tau_j^0 f_{i,t}^0 - f_{i}^0 \tau_j^0 &= \frac{1}{2} c_{j\alpha} c_{k\beta} \tau_{i,k}^0, \tag{5.2} \\
c_{j\alpha} \xi_{i,j}^1 - \frac{c_{i\alpha}}{2} (\tau_j^1 + f_j^0 \tau_j^0) &= \frac{c_{i\alpha}}{2} (f_j^0 \tau_j^0 + \frac{1}{2} c_{j\alpha} c_{k\beta} \tau_{i,k}^0), \\
c_{j\alpha} \tau_j^1 &= 0.
\end{align*}
\]
Group analysis of stochastic differential systems

The proof of this theorem is similar to that of Theorem 2.1. For the sake of brevity we omit the proof. However, one should notice that the Stratonovich differential obeys the standard chain rule [20]. This, in turn, leads to

\[ dB_\alpha = dB_\alpha + \frac{1}{2} \left[ \tau^0_t + f^0_j \tau^0_j + \varepsilon (\tau^1_t + f^0_j \tau^1_j + f^1_j \tau^0_j) \right] \circ dB_\alpha, \]

\[ dc^m_i = \left[ \xi^m_i + f^0_j \xi^m_j + \varepsilon f^1_j \xi^m_j \right] dt + \sqrt{\varepsilon} c_{\alpha \beta} c^m_i dB_\beta, \]

\[ d\tau^m_j = \left[ \tau^m_j + f^0_j \tau^m_j + \varepsilon f^1_j \tau^m_j \right] dt + \sqrt{\varepsilon} c_{\alpha \beta} \tau^m_j dB_\beta, \]

which is used in the proof of Theorem 2.2.

Let us define the vector fields

\[ D^m = \frac{\partial}{\partial t} + f^m_j \frac{\partial}{\partial x_j}, \quad X^m = \tau^m_t + \xi^m_k \frac{\partial}{\partial x_k}, \quad C_\alpha = c_{\alpha \beta} \frac{\partial}{\partial x_\beta}, \quad m = 0, 1, \] (5.4)

Eqs. (5.3) of the determining system can be rewritten now in the following form:

\[ [D^0, X^0] = (\tau^0_t + f^0_j \tau^0_j) D^0, \]

\[ [C_\alpha, X^0] = \frac{1}{2} (\tau^0_t + f^0_j \tau^0_j) C_\alpha, \]

\[ [D^0, X^1] = -[D^1, X^0] + (\tau^1_t + f^0_j \tau^1_j + f^1_j \tau^0_j) D^0 + (\tau^0_t + f^0_j \tau^0_j) D^1, \]

\[ [C_\alpha, X^1] = \frac{1}{2} (\tau^1_t + f^0_j \tau^1_j + f^1_j \tau^0_j) C_\alpha, \] (5.5)

where the expressions in the left-hand sides are the Lie brackets.

6 The relation between approximate symmetries of the Fokker-Planck equation and the Itô system

To every Itô stochastic ordinary differential equation there corresponds a deterministic partial differential equation called the Fokker-Planck equation. Its normalizable solution is the probability distribution of the solution to the stochastic differential equation (see [21] and [22]).
For the Itô system (4.1), the associate Fokker-Planck equation has the form

\[ F \equiv p_t + f_k^0 p_k + f_{k,k}^0 p + \varepsilon(f_{k,k,k}^1 + f_{k,k}^1 p - \frac{1}{2} c_{k,\alpha} c_{\alpha} p_{k,\alpha}) = 0. \]  

(6.1)

Here, we will discuss a correspondence between the approximate symmetries of the equations (4.1) and (6.1). Since the Fokker-Planck equation is a deterministic partial differential equation its approximate symmetries are found by using the method given in Section 2. Namely, we look for the approximate group generator (2.14)-(2.15) written in the form

\[ X = \tau^0(x, t, p) \frac{\partial}{\partial t} + \xi^0_i(x, t, p) \frac{\partial}{\partial x_i} + \Pi^0(x, t, p) \frac{\partial}{\partial p} + \zeta^0_t \frac{\partial}{\partial p_t} \]

+ \varepsilon \left[ \tau^1(x, t, p) \frac{\partial}{\partial t} + \xi^1_i(x, t, p) \frac{\partial}{\partial x_i} + \Pi^1(x, t, p) \frac{\partial}{\partial p} + \zeta^1_t \frac{\partial}{\partial p_t} + \zeta^1_i \frac{\partial}{\partial p_i} + \zeta^1_{ij} \frac{\partial}{\partial p_{ij}} \right],

and consider its second prolongation:

\[ X_{(2)} = \tau^0(x, t, p) \frac{\partial}{\partial t} + \xi^0_i(x, t, p) \frac{\partial}{\partial x_i} + \Pi^0(x, t, p) \frac{\partial}{\partial p} + \zeta^0_t \frac{\partial}{\partial p_t} + \zeta^0_i \frac{\partial}{\partial p_i} + \zeta^0_{ij} \frac{\partial}{\partial p_{ij}} \]

+ \varepsilon \left[ \tau^1(x, t, p) \frac{\partial}{\partial t} + \xi^1_i(x, t, p) \frac{\partial}{\partial x_i} + \Pi^1(x, t, p) \frac{\partial}{\partial p} + \zeta^1_t \frac{\partial}{\partial p_t} + \zeta^1_i \frac{\partial}{\partial p_i} + \zeta^1_{ij} \frac{\partial}{\partial p_{ij}} \right],

where

\[ \zeta^m_t = \Pi^m_t + p_t \Pi^m_p - p_k (\xi^m_{k,t} + p_t \xi^m_{k,p}) - p_t (\tau^m_t + p_t \tau^m_p), \]

\[ \zeta^m_i = \Pi^m_i + p_t \Pi^m_p - p_k (\xi^m_{k,i} + p_t \xi^m_{k,p}) - p_t (\tau^m_t + p_t \tau^m_p), \]

\[ \zeta^m_{ij} = D_j (\zeta^m_i) - p_{it} \tau^m_j - p_{ik} \xi^m_{k,j} + p_{i,k} \xi^m_{k,p} - p_{it} p_{j,\rho} \tau^m_{\rho}. \]

Here, \( m = 0, 1 \) and

\[ D_j = \frac{\partial}{\partial x_j} + p_j \frac{\partial}{\partial p}. \]

The approximate invariance criterion reads (cf. Eq. (2.17), and [14], Section 9.5.2)

\[ X_{(2)}(F) \big|_{F=\varepsilon} = o(\varepsilon). \]

It leads to the following determining system.

\[ c_{i,\alpha} c_{\alpha} \xi_{k,i}^0 + c_{k,\alpha} c_{j,\alpha} \xi_{k,j}^0 - c_{i,\alpha} c_{\alpha} (\tau^0_t + f^0_{i,x} \tau^0_x) \]

\[ + \varepsilon \left[ c_{i,\alpha} c_{\alpha} \xi_{k,i}^1 + c_{k,\alpha} c_{j,\alpha} \xi_{k,j}^1 - c_{i,\alpha} c_{\alpha} (\tau^1_t + f^1_{i,x} \tau^1_x + f^1_{i,x} \tau^1_x) \right] = 0, \]

(6.2)

\[ \xi_{k,t}^0 - (\tau^0 f_k^0)_{,t} + f_k^0 \xi_{k,i}^0 - c_{i,\alpha} c_{\alpha} (\tau^0_t + f^0_{i,x} \tau^0_x) \]

\[ + \varepsilon \left[ (\tau^1 + f_{x})_{,t} + f^1_{i,x} \xi_{k,i}^1 + f^0_{i,x} \xi_{k,i}^1 - c_{i,\alpha} c_{\alpha} (\tau^0_t + f^0_{i,x} \tau^0_x) \right] = 0, \]

(6.3)

\[-\frac{1}{2} c_{i,\alpha} c_{\alpha} \xi_{k,i}^0 - f^0_{i,k} \tau^0_x - f^1_{i,k} \tau^1_x - f^0_{i,k} \tau^0_x + c_{i,\alpha} c_{\alpha} \Pi^1_{,t} = 0, \]
Hence, we have just proven the following theorem.

In case of constant solutions where,

\[ \Pi_{\alpha}^{10} + f_{i}^{0} \Pi_{\alpha}^{10} + (\tau^{0} f_{i,\alpha}^{0})_{,\alpha} + f_{k,\alpha}^{0} \xi_{i}^{0} + f_{i,\alpha}^{0} \tau_{\alpha,m}^{0} \]

\[ + \varepsilon \left[ \Pi_{\alpha}^{11} + f_{i}^{0} \Pi_{\alpha}^{11} + f_{i}^{0} \Pi_{\alpha}^{10} - \frac{1}{2} c_{\alpha \alpha} c_{j \alpha} \Pi_{\alpha j}^{10} + (\tau^{0} f_{i,\alpha}^{1})_{,\alpha} + (\tau^{1} f_{i,\alpha}^{0})_{,\alpha} \right] (6.4) \]

and

\[ \Pi_{\alpha}^{2} + f_{k}^{0} \Pi^{2} + f_{k}^{0} \Pi_{\alpha}^{2} + \varepsilon \left( f_{k}^{1} \Pi_{\alpha}^{1} + f_{k}^{1} \Pi_{\alpha}^{2} - \frac{1}{2} c_{\alpha \alpha} c_{\alpha} \Pi_{\alpha \alpha}^{2} \right) = 0, \]

(6.5)

where

\[ \Pi^{0}(x, t, p) + \varepsilon \Pi^{1}(x, t, p) = (\Pi^{10}(x, t) + \varepsilon \Pi^{11}(x, t)) p + \Pi^{2}(x, t, \varepsilon), \]

\[ \tau^{m} = \tau^{m}(x, t), \quad \xi^{m}_{j} = \xi^{m}_{j}(x, t), \quad m = 0, 1. \]

Let us suppose that \( \xi^{m}(x, t) \) and \( \tau^{m}(x, t) \) satisfy (4.15)-(4.16). We now want to determine conditions under which they also satisfy (6.2)-(6.4). Equation (6.2) can be rewritten as

\[ c_{\alpha \alpha} T_{\alpha}^{0} + c_{\alpha \alpha} T_{\alpha}^{1} + \varepsilon \left( c_{\alpha \alpha} T_{\alpha}^{1} + c_{\alpha \alpha} T_{\alpha}^{1} \right) = 0, \]

where,

\[ T_{\alpha}^{0} = c_{j \alpha} \xi_{j,\alpha}^{0} - \frac{c_{\alpha \alpha}}{2} (\tau^{0}_{\alpha,\alpha} + f_{j}^{0} \tau^{0}_{j,\alpha}), \]

\[ T_{\alpha}^{1} = c_{j \alpha} \xi_{j,\alpha}^{1} - \frac{c_{\alpha \alpha}}{2} (\tau^{1}_{\alpha,\alpha} + f_{j}^{1} \tau^{0}_{j,\alpha} + f_{j}^{0} \tau^{1}_{j,\alpha}). \]

Since \( \xi(x, t)^{m} \) and \( \tau(x, t)^{m} \) satisfy (4.16), we have \( T_{\alpha}^{m} = 0, \quad m = 0, 1. \)

Differentiating (6.2) with respect to \( x_{i} \) and summing with (6.3) one obtains

\[ c_{\alpha \alpha} Q_{\alpha}^{0} = 0, \quad c_{\alpha \alpha} Q_{\alpha}^{1} = 0, \]

(6.6)

where

\[ Q^{0} = \Pi^{10} + \xi_{i,i}^{0} - f_{i}^{0} \tau_{i,i}^{0}, \quad Q^{1} = \Pi^{11} + \xi_{i,i}^{1} - f_{i}^{1} \tau_{i,i}^{0} - f_{i}^{0} \tau_{i,i}^{1}. \]

Differentiating (6.2) with respect to \( x_{k} \) and \( x_{i} \), then differentiating (6.3) with respect to \( x_{k} \) and finally summing the resulting equations with equation (6.4) one obtains

\[ Q_{k}^{0} + f_{k}^{0} Q_{k}^{0} = 0, \quad Q_{k}^{1} + f_{k}^{0} Q_{k}^{1} + f_{k}^{1} Q_{k}^{1} = 0. \]

(6.7)

Solutions to Equations (6.6) and (6.7) provide a relation between the approximate symmetries of the Fokker-Planck equation and the approximate symmetries of Itô system. In case of constant solutions \( C_{1} \) and \( C_{2} \) we find

\[ \Pi^{10} = -\xi_{i,i}^{0} + f_{i}^{0} \tau_{i,i}^{0} + C_{1}, \quad \Pi^{11} = -\xi_{i,i}^{1} + f_{i}^{1} \tau_{i,i}^{0} + f_{i}^{0} \tau_{i,i}^{1} + C_{2}. \]

Hence, we have just proven the following theorem.
Theorem 6.1. Let
\[
X = \tau^0 \frac{\partial}{\partial t} + \xi^0_j \frac{\partial}{\partial x_j} + \varepsilon \left( \tau^1 \frac{\partial}{\partial t} + \xi^1_j \frac{\partial}{\partial x_j} \right)
\]
be the generator of the approximate symmetry of the Itô system (4.1). Then
\[
Y = X + \left[ C_1 - \xi^0_{i,a} + f^0_i \tau^0_{i,a} + \varepsilon (C_2 - \xi^1_{i,a} + f^1_i \tau^1_{i,a} + f^0_i \tau^1_{i,a}) \right] p \frac{\partial}{\partial p}.
\]
Likewise, one can easily show that if \(Y\) is an approximate symmetry of the Fokker-Planck equation (6.1) then \(X\) is an approximate symmetry of the Itô dynamical system (4.1).

7 Approximate conservation laws for stochastic dynamical systems

Conserved quantities of Stratonovich dynamical systems were considered in [1] and [2] without recourse to Hamiltonian formulation. Here we give an alternative theorem for approximate conservation laws. First we recall the necessary notation from the exterior differential calculus.

7.1 Preliminaries on differential forms

A differential form of degree \(p\) (or a \(p\)-form) is given by
\[
\omega = \sum_{i_1<\ldots<i_p} a_{i_1\ldots i_p}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_p},
\]
where \(x = (x^1, \ldots, x^n) \in \mathbb{R}^n\), \(dx = (dx^1, \ldots, dx^n) \in \mathbb{R}^n\), and \(a_{i_1\ldots i_p}(x)\) are continuously differentiable functions. The summation is extended over all values \(i_1, \ldots, i_p = 1, \ldots, n\) such that \(i_1 < \cdots < i_p\). The exterior differential calculus is defined by the law of exterior multiplication
\[
d x^i \wedge dx^j = -dx^j \wedge dx^i \quad \text{(in particular, } dx^i \wedge dx^i = 0),
\]
and by the exterior differentiation
\[
d \omega = \sum_{i_1<\ldots<i_p} \sum_{j=1}^n \frac{\partial a_{i_1\ldots i_p}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}.
\]
Accordingly, the differential \(d\omega\) is a \((p + 1)\)-form. The exterior differentiation and multiplication of forms obey the following rules:
\[
d^2 \omega \equiv d(d\omega) = 0,
\]
\[ \omega \wedge \eta = (-1)^{pq} \eta \wedge \omega, \]
\[ d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta, \]
where \( \omega \) is a \( p \)-form and \( \eta \) a \( q \)-form. If \( p = n \), then any \( n \)-form is written \( \omega = a(x)dx^1 \wedge \cdots \wedge dx^n \), and its integral is defined by
\[ \int \omega = \int a(x)dx^1 \cdots dx^n. \]

Let \( \omega \) be a differential form of degree \( p \) and \( X \) is a vector field. Then the interior product | of \( X \) by \( \omega \) yields a differential form \( \omega' \) of degree \( p - 1 \):
\[ \omega' = X|\omega. \]

In order to explain the meaning of the interior product, consider the case \( p = 1 \). Let \( X \) be a vector field:
\[ X = \eta_j \frac{\partial}{\partial x_j} \]
and \( \omega \) be the following simple 1-form:
\[ \omega = dx_k. \]

Then
\[ X|\omega = \eta_j \frac{\partial}{\partial x_j} |dx_k = \eta_j \delta_{jk} = \eta_k. \]

Accordingly, the interior product of a 2-form with two vector fields is defined as follows. Let
\[ X^1 = \eta_1^1 \frac{\partial}{\partial x_1} + \eta_1^2 \frac{\partial}{\partial x_2}, \quad X^2 = \eta_2^1 \frac{\partial}{\partial x_1} + \eta_2^2 \frac{\partial}{\partial x_2} \]
and
\[ \Omega = dx_1 \wedge dx_2. \]

Then
\[ X^1|X^2|\Omega = X^1|((\eta_2^2 dx_2 - \eta_1^2 dx_1) = -\eta_1^1 \eta_2^2 + \eta_1^2 \eta_2^1 \]
which can be rewritten as
\[ X^1|X^2|\Omega = - \det \begin{bmatrix} \eta_1^1 & \eta_1^2 \\ \eta_2^1 & \eta_2^2 \end{bmatrix}. \]

Let us consider now \( n \)-linearly independent vector fields
\[ X^1 = \eta_1^j \frac{\partial}{\partial x_j}, \ldots, X^n = \eta_n^j \frac{\partial}{\partial x_j} \]
and the following $n$-form:

$$\Omega = dx_1 \wedge \cdots \wedge dx_n.$$ 

Then

$$X^1 | \cdots | X^n | \Omega = - \det \begin{vmatrix} \eta_1^1 & \eta_1^2 & \cdots & \eta_1^n \\ \eta_2^1 & \eta_2^2 & \cdots & \eta_2^n \\ \vdots & \vdots & \ddots & \vdots \\ \eta_n^1 & \eta_n^2 & \cdots & \eta_n^n \end{vmatrix}.$$ 

### 7.2 Conservation laws

**Definition 7.1.** We call $I^0(x, t) + \varepsilon I^1(x, t)$ an approximate conserved quantity if

$$I^0(x, t) + \varepsilon I^1(x, t) \approx C, \quad C = \text{const.},$$

on the sample paths of the stochastic dynamical system.

The Itô differential of the conserved quantity $I(x, t)$ can be easily obtained from (4.7), namely

$$dI^0 + \varepsilon dI^1 = \left[ I^0_t + f_j^0 I^0_j + \varepsilon (I^1_t + f_j^0 I^1_j + f_j^1 I^0_j) + \frac{\sqrt{\varepsilon}}{2} (c_{ja} c_{ka} I^0_{jk} + \varepsilon c_{ja} c_{ka} I^1_{jk}) \right] dt + \sqrt{\varepsilon} (c_{ja} I^0_j + \varepsilon c_{ja} I^1_j) dB_\alpha = O(\varepsilon^2).$$

This leads to the following partial differential equations:

$$I^0_t + f_j^0 I^0_j = 0, \quad c_{ja} I^0_j = 0, \quad I^1_t + f_j^0 I^1_j = - f_j^1 I^0_j, \quad c_{ja} I^1_j = 0$$

(7.1) for an approximate conserved quantity to satisfy. One can easily show that an approximate conserved quantity of the Stratonovich dynamical system (5.1) should also satisfy (7.1).

Using the vector fields given in (5.4), equation (7.1) can be rewritten as

$$D^0(I^0) = 0, \quad C^0_\alpha(I^0) = 0,$$

(7.2)

$$D^0(I^1) = - D^1(I^0), \quad C^0_\alpha(I^1) = 0.$$  

(7.3)

**Theorem 7.1.** Suppose that Itô dynamical system (4.1) admits a stable approximate symmetry vector field of the form

$$X = X_0 + \varepsilon X_1 = X^0_t + \xi^0_i \frac{\partial}{\partial x_i} + \varepsilon \left( \tau^i_t \frac{\partial}{\partial t} + \xi^1_i \frac{\partial}{\partial x_i} \right)$$

Then an approximate conservation law $I(x, t) = I^0(x, t) + \varepsilon I^1(x, t)$ can be found as

$$I^0(x, t) = \xi^0_{x_i} - f^0_{x_i} \tau^0_t + P^0,$$

$$I^1(x, t) = \xi^1_{x_i} - f^1_{x_i} \tau^0_t - f^0_{x_i} \tau^1_t + P^1.$$  

(7.4)

where $P^0$ and $P^1$ satisfy Equations (6.3) and (6.4).
Proof. It follows immediately from the comparison between the Equation (7.1) and
Equations (6.6)-(6.7).

Theorem 7.2. Let
\[ \text{div } D^m = 0, \quad (m = 0, 1) \]
and let
\[ X_1 = X_1^0 + \varepsilon X_1^1, \quad \ldots \quad X_n = X_n^0 + \varepsilon X_n^1 \]
be the linearly independent symmetry vector fields satisfying the properties
\[ \tau_t^0 + f_j^0 \tau_{ij}^0 = 0, \quad \tau_t^1 + f_j^0 \tau_{ij}^1 + f_j^1 \tau_{ij}^0 = 0. \tag{7.5} \]
Then the approximate conserved quantity \( I^0(x, t) + \varepsilon I^1(x, t) \) can be obtained from the
approximate symmetries as follows:
\[ I^0(x, t) = X_1^0[X_2^0] \cdots [X_n^0] \Omega, \]
\[ I^1(x, t) = X_1^1[X_2^0] \cdots [X_n^0] \Omega + X_1^0[X_2^1] \cdots [X_n^0] \Omega \tag{7.6} \]
\[ + \cdots + X_1^0[X_2^0] \cdots [X_n^1] \Omega, \]
where \( \llbracket \) is the interior product and \( \Omega = dx_1 \wedge \cdots \wedge dx_n \) is the volume form.

Proof. Let us calculate the Lie derivative of \( I^0 \) with respect to \( D^0 \) to obtain
\[ D^0(I^0) \equiv \mathcal{L}_{D^0} I^0 = (\mathcal{L}_{D^0} X_1^0)[X_2^0] \cdots [X_n^0] \Omega + \cdots
\]
\[ + X_1^0(\mathcal{L}_{D^0} X_2^0) \cdots [X_n^0] \Omega + X_1^0[X_2^1] \cdots [X_n^0] \Omega \tag{7.7} \]
Using (7.5) in (5.5) we find
\[ [D^0, X_j^0] \equiv \mathcal{L}_{D^0} X_j^0 = 0, \quad (j = 1, \ldots, n). \]
Since
\[ \mathcal{L}_{D^0} \Omega = \text{div } D^0 \Omega \]
and \( \text{div } D^0 = 0 \), one has \( \mathcal{L}_{D^0} \Omega = 0 \). Therefore all the terms on the right hand side of
(7.7) vanish. We now calculate the Lie derivative of \( I^0 \) with respect to \( C_0 \) to obtain
\[ C_0(I^0) \equiv \mathcal{L}_{C_0} I^0 = (\mathcal{L}_{C_0} X_1^0)[X_2^0] \cdots [X_n^0] \Omega + \cdots
\]
\[ + X_1^0(\mathcal{L}_{C_0} X_2^0) \cdots [X_n^0] \Omega + X_1^0[X_2^1] \cdots [X_n^0] \Omega \tag{7.8} \]
Using (7.5) in (5.5) we find
\[ [C_0, X_j^0] \equiv \mathcal{L}_{C_0} X_j^0 = 0, \quad (j = 1, \ldots, n). \]
Furthermore, \( \mathcal{L}_{C_n} \Omega = 0 \). Therefore all the terms on the right hand side of (7.8) also vanish. Hence, Eqs. (7.2) hold for
\[
X^0 \big| X^0_2 \big| \cdots \big| X^0_n \big| \Omega.
\]

Similar calculations for \( I^1 \) given in (7.6) lead to the equation
\[
\mathcal{L}_{D^0} I^1 = -(\mathcal{L}_{D^1} X^0_1) \big| X^0_2 \big| \cdots \big| X^0_n \big| \Omega
- X^0 \big| (\mathcal{L}_{D^1} X^0_2) \big| \cdots \big| X^0_n \big| \Omega + \cdots - X^0_1 \big| X^0_2 \big| \cdots \big| (\mathcal{L}_{D^1} X^0_n) \big| \Omega.
\]
It can be rewritten as
\[
\mathcal{L}_{D^0} I^1 = -\mathcal{L}_{D^1} [X^0_1 X^0_2 \cdots X^0_n] \Omega.
\]
Since
\[
X^0 \big| X^0_2 \big| \cdots \big| X^0_n \big| \Omega = I^0
\]
we arrive at the equation
\[
\mathcal{L}_{D^0} I^1 = -\mathcal{L}_{D^1} I^0.
\]
It can be easily shown that
\[
\mathcal{L}_{C_n} I^1 = 0.
\]
Hence \( I^1 \) given in (7.6) satisfies (7.3). This completes the proof.

### 8 An Application

Let us consider the following stochastic dynamical system:
\[
dx_1 = x_2 dt, \quad dx_2 = (-x_1 + \varepsilon f(x_1, x_2)) dt + \sqrt{\varepsilon} dB. \tag{8.1}
\]
For this problem, we have
\[
f^0_1 = x_2, \quad f^0_2 = -x_1, \quad f^1_1 = 0, \quad f^1_2 = f(x_1, x_2), \quad c_{11} = c_{12} = c_{22} = 0, \quad c_{21} = 1.
\]
The determining equations (5.2) for Eqs. (8.1) are written
\[
\xi^0_{1,t} + x_2 \xi^0_{1,1} - x_1 \xi^0_{1,2} - \xi^0_2 - x^0_2 \tau^0_{1,t} - x^0_2 \tau^0_{1,1} = 0,
\]
\[
\xi^0_{2,t} + x_2 \xi^0_{2,1} - x_1 \xi^0_{2,2} + \xi^0_1 + x_1 \tau^0_{1,t} + x_1 x_2 \tau^0_{1,1} = 0,
\]
\[
\tau^0_{1,2} = 0, \quad \xi^0_{1,1} = 0, \quad \xi^0_{2,2} - \frac{1}{2} \left( \tau^0_{1,t} + x_2 \tau^0_{1,1} \right) = 0
\]
and yield
\[
\tau^0 = C_1, \quad \xi^0_1 = C_2 \cos t + C_3 \sin t, \quad \xi^0_2 = C_3 \cos t - C_2 \sin t.
\]
Now the determining equations (5.3) become
\[
\begin{align*}
\xi_{1,t}^1 + x_2\xi_{1,1}^1 - x_1\xi_{1,2}^1 - \xi_2^1 - x_2\tau_{t,1}^1 - x_2^2\tau_{1,1}^1 &= 0, \\
\xi_{2,t}^1 + x_2\xi_{2,1}^1 - x_1\xi_{1,2}^1 + \xi_1^1 + x_1\tau_{t,1}^1 + x_1x_2\tau_{1,1}^1 &= \xi_1^0 f_{,1} + \xi_2^0 f_{,2}, \\
\tau_{t,2}^1 &= 0, \quad \xi_{1,2}^1 = 0, \quad \xi_{1,2}^1 - \frac{1}{2}(\tau_{t,1}^1 + x_2\tau_{1,1}^1) = 0.
\end{align*}
\]
It follows:
\[
\begin{align*}
\xi_{1}^1 &= \frac{3}{2}\tau_{1}^1 x_1 + Z(t), \\
\xi_{2}^1 &= \frac{1}{2}\tau_{1}^1 x_2 + \frac{3}{2}\tau_{1}^1 x_1 + Z(t),
\end{align*}
\]
where \(\tau_{1}^1\) satisfies the equation
\[
\begin{align*}
x_1\frac{d}{dt}\left(\frac{3}{2}\tau_{1}^1 + 2\tau_{1}^1\right) + 2\tau_{1}^1 x_2 + \ddot{Z} + Z &= (C_2 \cos t + C_3 \sin t)f_{,1} + (C_3 \cos t - C_2 \sin t)f_{,2},
\end{align*}
\] (8.2)
where the dot denotes the differentiation with respect to \(t\). It is manifest from Eq. (8.2) that we have to distinguish the following two cases:

I) \(f(x_1, x_2) = Ax_1 + Bx_2 + Cx_1^2 + Dx_1x_2 + Ex_2^2\),

II) \(f(x_1, x_2)\) is an arbitrary function.

The first case involves the following subcases:

I(i). \(f(x_1, x_2) = Ax_1 + Bx_2 + C(x_1^2 - x_2^2)\).

Then the approximate symmetry vector fields are:
\[
X_1 = \cos t \frac{\partial}{\partial x_1} - \sin t \frac{\partial}{\partial x_2} + \varepsilon \left[ \left( At \sin t - Bt \cos t - \frac{3}{2} Cx_1 \cos t \right) \frac{\partial}{\partial x_1} \right],
\]
\[
X_2 = \sin t \frac{\partial}{\partial x_1} + \cos t \frac{\partial}{\partial x_2} + \varepsilon \left[ \left( At \cos t + Bt \sin t - \frac{3}{2} Cx_1 \sin t \right) \frac{\partial}{\partial x_1} \right],
\]
\[
X_3 = \varepsilon \left( \cos t \frac{\partial}{\partial x_1} - \sin t \frac{\partial}{\partial x_2} \right), \quad X_4 = \varepsilon \left( \sin t \frac{\partial}{\partial x_1} + \cos t \frac{\partial}{\partial x_2} \right).
\] (8.5)

I(ii). \(f(x_1, x_2) = Ax_1 + Bx_2 + Dx_1x_2 + C(x_1^2 - x_2^2)\). The approximate symmetry vector fields are \(X_2\), \(X_3\) and \(X_4\).

I(iii). \(f(x_1, x_2) = Ax_1 + Bx_2 + Cx_1^2 + Ex_2^2\). The approximate symmetry vector fields are \(X_1\), \(X_3\) and \(X_4\).
Figure 3.1: \( f = Ax_1 + Bx_2 + C(x_1^2 - x_2^2) \), where \( A = -0.1 \), \( B = -0.4 \), \( C = 1 \) and \( \varepsilon = 0.2 \). Here: a) \( dB(t) \), b) \( x_1(t) \), c) \( x_2(t) \).
Figure 3.2: $f = Ax_1 + Bx_2 + Cx_1^2 + Dx_1x_2 + Ex_2^2$, where $A = -1$, $B = -1$, $C = 1$, $E = 1$ and $\varepsilon = 0.2$. Here: a) $dB(t)$, b) $x_1(t)$, c) $x_2(t)$.
I(iv). $f(x_1, x_2) = Ax_1 + Bx_2 + Cx_1^2 + Dx_1x_2 + Ex_2^2$. The approximate symmetry vector fields are $X_3$ and $X_4$.

In Case II, i.e. when $f(x_1, x_2)$ is an arbitrary function, it follows from Eq. (8.2) that the approximate symmetries are $X_3$ and $X_4$.

Theorem 5.1 furnishes the following approximate conserved quantity for the system (8.1) in the case I(i):

$$I^0 = X^0_1 |X^0_2| \Omega = 1, \quad \Omega = dx_1 \wedge dx_2,$$

$$I^1 = X^1_1 |X^0_2| \Omega + X^0_1 |X^1_2| \Omega = 3Cx_1 + (B - A)t \cos(2t) - (A + B) \sin(2t).$$

To illustrate a typical random disturbance and solutions to stochastic differential equations, we have numerically integrated the stochastic differential equation (8.1) for cases I(i) and I(iv). The results are plotted in Fig. 3.1 and Fig. 3.2, respectively.

Acknowledgements

The work was done during the visit of Gazanfer Ünal to ALGA in Spring of 2004. G. Ünal acknowledges the financial support provided by ALGA.
Bibliography

[1] T. Misawa, “New conserved quantities derived form symmetry for stochastic dy-


[5] G. Ünal, “Symmetries of Itô and Stratonovich dynamical systems and their con-


Archives of ALGA

Brief facts about the centre ALGA: Advances in Lie Group Analysis

ALGA at Blekinge Institute of Technology, Sweden, is an international research and educational centre aimed at producing new knowledge of Lie group analysis of differential equations and enhancing the understanding of the classical results and modern developments.

The main objectives of ALGA are:

- To make available to a wide audience the classical heritage in group analysis and to teach courses in Lie group analysis as well as new mathematical programs based on the philosophy of group analysis.

- To advance studies in modern group analysis, differential equations and non-linear mathematical modelling and to implement a database containing all the latest information in this field.

For more information, contact the director of ALGA, Professor N.H. Ibragimov.
E-mail: nib@bth.se
Homepage: www.bth.se/alga
Address: ALGA, Blekinge Institute of Technology, S-371 79 Karlskrona, Sweden.

Aims and Scope of Archives

**Aim:** The aim of the Archives is to provide an international forum for classical and modern group analysis by means of rapid communication of new results, review articles and publications of historical heritage. The publications are related to the activities of ALGA.

**Scope:** The scope of Archives encompasses Lie group analysis with a focus on non-linear differential equations and mathematical models in science and engineering.